

For foreseeable future, R comm ring w/ 1

Review

R is Noetherian if it satisfies the ACC on ideals,

given $I_1 \subseteq I_2 \subseteq \dots \exists k$ s.t. $I_k = I_{k+1} = \dots$

(DCC called Artinian,
much stronger! DCC \rightarrow ACC)

Thm

1. R Noeth \Rightarrow R/I Noeth. (Cor Thm)

2. TFAE

a. R Noeth

b. Every nonempty set of ideals has max elem.

c. Every ideal f.g.

Hilbert Basis Thm R Noeth \Rightarrow R[x] is

Proof

Suppose $I \subseteq R[x]$ not f.g. Construct $P_1(x), P_2(x), P_3(x)$ etc. in I such that P_i has minimal degree among elems in $I - (P_1(x), P_2(x), \dots, P_{i-1}(x))$

Let a_i be leading coef of $P_i(x)$

Def $L = (a_1, a_2, \dots) \subseteq R$, so L is f.g., say $L = (a_1, a_2, \dots, a_N)$

Thus:

$a_{N+1} = \sum r_i a_i$ Consider:

$f(x) = \sum_{i=1}^N r_i P_i(x)$ deg $P_{N+1}(x) - \text{deg } P_i(x)$

Note that

1. deg $f(x) = \text{deg } P_{N+1}(x)$
2. $f(x)$ has lead coef a_{N+1} and $f(x) \in (P_1(x), P_2(x), \dots, P_N(x))$
3. Thus $P_{N+1}(x) - f(x) \notin (P_1(x), P_2(x), \dots, P_N(x))$ and has smaller deg. \neq

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Special Case

$R = k$, $k[x_1, x_2, \dots, x_n]$ is a k -algebra. (vs w/ mult or comm ring w/ scalars)

Prop F.g. k -algebras \leftrightarrow homo images of $k[x_1, \dots, x_n]$ $\leftrightarrow k[x_1, \dots, x_n]/(I)$

Goal Relate Comm Alg \leftrightarrow geometry

Def. \forall Let k be field, $A^n = \{(a_1, \dots, a_n) \mid a_i \in k\}$ affine n -space.

- like vector space with origin forgotten
- can be done axiomatically

Def. $k[A^n] = k[x_1, x_2, \dots, x_n]$ coordinate ring of A^n , functions

Def. Let $S \subseteq k[A^n]$ Define

$$Z(S) = \{(a_1, a_2, \dots, a_n) \mid f(a_1, a_2, \dots, a_n) = 0 \forall f \in S\}$$

= zero set.

Def. $V \subseteq A^n$ is an (affine) algebraic variety if $V = Z(S)$ some $S \subseteq k[A^n]$

Remarks

1. HBT says S is finite WLOG

2. AKA algebraic set, algebraic variety means irreducible algebraic variety - not union of two proper


EX $S = \{xy\} \subseteq k[A^2]$



$V =$ union of 2, not irreducible.

algebraic

Examples of Varieties

- 1. $\mathbb{A}^n \quad S = \{\emptyset\}, \emptyset, \{S = \{1\}\}$
- 2. Varieties in $\mathbb{A}^1 \leftrightarrow$ finite sets of points or \mathbb{R} .
- 3. Any Finite set in \mathbb{A}^n (exercise)
- 4. IF $|S|=1$ then $Z(S) = Z(p(x))$ is called a hypersurface.
- 5. $S = \{z - x^2 - y^2\}$  not smooth.

Elementary Properties

- 1. IF $(S) = I$ then $Z(S) = Z(I)$, so WLOG take S to be an ideal, HBT says any variety is \cap of finite # hypersurfaces
- 2. $Z(S) \cap Z(T) = Z(S \cup T)$, also $\bigcap_i Z(S_i)$
- 3. $Z(I) \cup Z(J) = Z(I \cap J)$
- 4. $I \subseteq J \rightarrow Z(J) \subseteq Z(I)$

Warning $Z: \text{Ideals in } k[\mathbb{A}^n] \rightarrow \text{alg varieties}$ is not 1-1.

EX $Z(x) = Z(x^2) = \{x=0\}$

but $(x^2) \subsetneq (x)$.

Going in Reverse

Let $A \subseteq A^n$. Def: $\mathcal{I}(A) = \{p(x) \in R[A^n] \mid p(a_1, \dots, a_n) = 0 \forall (a_1, \dots, a_n) \in A\}$

Rank $\mathcal{I}(A)$ is clearly an ideal, and is the largest ideal of functions which are zero on A .

Ex 1 $I = (x) \rightarrow Z(I) = \{x=0\} \Rightarrow \mathcal{I}(\{x=0\}) = (x)$
 $J = (x^2) \rightarrow Z(J) = \{x=0\}$

Ex Recall given $(a_1, a_2, \dots, a_n) = \bar{a}$, $eV_{\bar{a}}: R[x_1, x_2, \dots, x_n] \rightarrow R$.

Thus

$$\mathcal{I}((a_1, a_2, \dots, a_n)) = \text{Ker } eV_{\bar{a}} \text{ is maximal ideal} \\ = (x_1 - a_1, x_2 - a_2, \dots, x_n - a_n)$$

Rank Variety $A^n \rightarrow \text{coord ring } R[A^n] \rightarrow \{\text{max ideals}\}$
 $\cong A^n$

Easy Properties Let $A, B \subseteq A^n$.

- $A \subseteq B \Rightarrow \mathcal{I}(B) \subseteq \mathcal{I}(A)$
- $\mathcal{I}(A \cup B) = \mathcal{I}(A) \cap \mathcal{I}(B)$
- $\mathcal{I}(\emptyset) = R[x_1, \dots, x_n]$ $\mathcal{I}(A^n) \neq 0$ if R is infinite!
Ex $R = \mathbb{F}^2 = A^1$ $x^2 + 1 \in \mathcal{I}(A^1)$
 R^∞ , polys can't have only many roots
- $A \subseteq Z(\mathcal{I}(A))$ ~~\mathcal{I}~~ $J \subseteq \mathcal{I}(Z(J))$
- Suppose $V = Z(I)$ an alg variety. Then $V = Z(\mathcal{I}(V))$.
Suppose $I = \mathcal{I}(A)$. Then $\mathcal{I}(Z(I)) = I$
i.e. $ZIZ = Z$, $IZI = I$.

Remark Thus Z, \mathcal{I} bijections varieties \leftrightarrow ideals of form $\mathcal{I}(V)$ ($R = \mathbb{R}$ radical ideal (c))

Def. Let $V \subseteq \mathbb{A}^n$ be alg variety, $k[\mathbb{A}^n] / \mathcal{I}(V) := k[V]$ is the coordinate ring of V .

Remark. Think of $k[V]$ as polynomials on V , i.e. $\overline{P(X)} = \overline{B(X)}$ iff $P|_V = B|_V$

Maps between Alg Varieties

Def. Let $V \subseteq \mathbb{A}^n, W \subseteq \mathbb{A}^m$ be varieties.

$\psi: V \rightarrow W$ is a regular map if $\exists \psi_i, \psi_j, \psi_m \in k[X_1, \dots, X_n]$ s.t.

$$\psi((a_1, a_2, a_3)) = (\psi_1(a_1, a_2, a_3), \dots, \psi_m(a_1, a_2, a_3))$$
$$\forall (a_1, a_2, a_3) \in V$$

i.e. ψ is a polynomial map. WARNING \cong requires ψ 1-1, onto and ψ^{-1} is regular

Exercise Let ψ be as above. Then

$$\tilde{\psi}: k[W] \rightarrow k[V], \quad \tilde{\psi}(f) = f \circ \psi$$

is well defined alg hom.

Thm Let $V \subseteq \mathbb{A}^n, W \subseteq \mathbb{A}^m$ alg varieties \exists bijection

$$\left\{ \begin{array}{l} \text{regular maps} \\ V \rightarrow W \end{array} \right\} \leftrightarrow \left\{ \begin{array}{l} k\text{-alg hom} \\ k[W] \rightarrow k[V] \end{array} \right\} \quad \text{that is}$$

- 1. Contravariant functor
- 2. "Eq of categories"

Remarks

1. Can think of regular maps $V \rightarrow W$ (geometry) purely algebraically

2. Prop Let $\psi: V \rightarrow W$ map of varieties

$$\psi \text{ is regular} \iff \forall f \in R[W] \exists \varphi \in R[V]$$

In particular $R[V]$ is WD in d. of embedding

3. Groebner Bases, powerful computing tools available