Lecture 1

Category Theory

Def A *category* $\mathcal{C}$ consists of a class of objects and sets of morphisms between the objects. For each $A, B \in \mathcal{C}$ $\exists$ a set $Hom_{\mathcal{C}}(A,B)$ of morphisms from A to B such that

1. There is law of composition $Hom_{\mathcal{C}}(A,B) \times Hom_{\mathcal{C}}(B,C) \to Hom_{\mathcal{C}}(A,B)$
   $(f,g) \longrightarrow gf$

2. $A \neq C$ or $B \neq D$ then $Hom_{\mathcal{C}}(A,B)$ and $Hom_{\mathcal{C}}(C,D)$ are disjoint.

3. Composition is associative.

4. For each $A \in \mathcal{C}$ $\exists$ $1_A \in Hom_{\mathcal{C}}(A,A)$ such that

   $$1_A f = f \quad \forall f \in Hom_{\mathcal{C}}(B,A), \qquad f 1_A = f \quad \forall f \in Hom_{\mathcal{C}}(A,B)$$

   identity morphism

Remarks

1. Class $\neq$ set, small category means objects are a set.

2. Maps $A \xrightarrow{f} B$ also called arrows. Note that objects need not have elements, so "composition" is formal
   concrete category

3. $Hom_{\mathcal{C}}(A,A)$ are endomorphisms. $f \in Hom_{\mathcal{C}}(A,B)$ is an $\cong$ if $\exists$ $g \in Hom_{\mathcal{C}}(B,A)$ with $gf = 1_A$, $fg = 1_B$

4. Subcategory makes sense $\mathcal{D} \subseteq \mathcal{C}$, may be fewer objects &/or fewer maps

Examples of categories

1. Set = cat of all sets. Hom(A,B) is all functions from A to B.

2. Grp = cat of all groups. Hom(G,H) = all group homomorphisms.
   subcategory Ab abelian groups.
   called <u>full subcategory</u>

3. Ring = cat all rings w/ 1, morphisms ring homos sending $1 \mapsto 1$
   CRing = comm rings w/ 1

4. Top = cat of all topological spaces with continuous functions.

5. R a ring, R-mod = all left R-modules w/ module homomorphisms.

   <u>Rmk</u> $\mathrm{Hom}_R(M,N)$ is an abelian group, kernels & cokernels exist.
   Say R-mod is an <u>abelian category</u>.

6. Let G be a group. Define a category w/ 1 object $*$ and
   $\mathrm{Hom}(*,*) = G$.
   "Group is category w/ one object and all maps are $\cong$'s"
   Monoid is a category w/ 1 object.

7. Consider R-mod. Say a map $f: M \to N$ factors through a projective if $\exists$ projective module P with

$$\begin{tikzcd} & P \arrow[dr] & \\ M \arrow[ur] \arrow[rr, "f"'] & & N \end{tikzcd}$$

Define $\underline{\mathrm{Hom}}_R(M,N) = \mathrm{Hom}_R(M,N)/\sim$

<u>Stable category</u>. st-mod
- Not full sub
- Not abelian cat

Functors

Def Let $\mathcal{C}$ & $\mathcal{D}$ be categories. $\mathcal{F}$ is a <u>covariant functor</u> from $\mathcal{C}$ to $\mathcal{D}$ if

1. For every $A \in \mathcal{C}$, $\mathcal{F}A \in \mathcal{D}$.
2. For every $f \in Hom_{\mathcal{C}}(A,B)$ we have $\mathcal{F}(f) \in Hom_{\mathcal{D}}(\mathcal{F}A, \mathcal{F}B)$ such that:
   - $\mathcal{F}(gf) = \mathcal{F}(g)\mathcal{F}(f)$
   - $\mathcal{F}(1_A) = 1_{\mathcal{F}A}$

Say $\mathcal{F}$ is <u>contravariant</u> if it reverses arrows:

$$f \in Hom_{\mathcal{C}}(A,B) \text{ then } \mathcal{F}(f) \in Hom_{\mathcal{D}}(\mathcal{F}B, \mathcal{F}A)$$

$$\mathcal{F}(gf) = \mathcal{F}(f)\mathcal{F}(g)$$

<u>Picture</u>

covariant:

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[rr, bend left, "gf"] & B \arrow[r, "g"] & C \end{tikzcd}$$

$$\begin{tikzcd} \mathcal{F}A \arrow[r, "\mathcal{F}(f)"] \arrow[rr, bend right, "\mathcal{F}(gf)"'] & \mathcal{F}(B) \arrow[r, "\mathcal{F}(g)"] & \mathcal{F}C \end{tikzcd}$$

contravariant:

$$A \xrightarrow{f} B$$

$$\mathcal{F}B \xrightarrow{\mathcal{F}(f)} \mathcal{F}A$$

Examples of functors

1. inclusion of a subcategory into a category.

2. Forgetful functor: $Grp \to Set$, etc..

3. Abelianization: $Grp \longrightarrow Ab$

   objects $\quad G \longrightarrow G/G'$

   maps $\quad \varphi: G \longrightarrow H$

   $\quad \bar{\varphi}: gG' = \varphi(g)H'$ $\qquad$ check well-def & axioms

4. Let R be a ring, N a left R-module. Then recall that $Hom_R({}_RM, {}_RN)$ is an abelian group

   Also, $\psi: M_1 \to M_2$ induces a map $\tilde{\psi}: Hom_R(M_2, N) \to Hom_R(M_1, N)$

   Thus $Hom_R(-, {}_RN)$ is a contravariant functor from R-mod to Ab

   Similarly $Hom_R({}_RN, -)$ is a covariant functor.

5. Let ${}_SM_R$ be an S-R bimodule. Recall

   ${}_SM_R \otimes_R {}_RN$ is a left S-mod

   Thus $\quad F: N_1 \to N_2$ get $1 \otimes F: M \otimes N_1 \to M \otimes N_2$

   So $\quad {}_SM_R \otimes_R - : R\text{-mod} \to S\text{-mod}$ covariant

Ex Let $V \in$ fd Vec. Then $v \in V$ we have $V^{**} =$ linear maps $V^* \to k$
$ev_v \in V^{**}$

$$V \xrightarrow{\cong} V^{**}$$
$$v \to ev_v : f \to f(v) \qquad \text{say nat} \cong$$

Thus $D^2 : V \to V^{**}$ is a functor fd-Vec to itself

$D^2 \psi \qquad \psi : V \to W$

$$D^2\psi : W^{**} \to V^{**}$$
$$D^2\psi(Ev_v) = E_{\psi(v)}$$

Def $\mathcal{F}$ is <u>faithful</u> if $\mathcal{F} : Hom(A,B) \to Hom(\mathcal{F}A, \mathcal{F}B)$
is injective $\forall A, B$

<u>full</u> " " surjective

<u>Rmk</u> Opp category reduces study of functors to one or other.

Natural Transformations

Motivation $V \in$ Fd VS/$\mathbb{R}$. Then $V \cong V^*$, must pick a basis.

But $v \to Ev_v$ is $\cong$ $V \to V^{**}$ which is natural.

Def Let $\mathcal{F}, \mathcal{G}$ be covariant functors $\mathcal{C} \to \mathcal{D}$.

A natural transformation from $\mathcal{F}$ to $\mathcal{G}$ is a map η that assigns to each object $A \in \mathcal{C}$ a morphism

$$\eta_A \in \mathrm{Hom}_{\mathcal{D}}(\mathcal{F}A, \mathcal{G}A) \text{ s.t.}$$

$\forall A, B \in \mathcal{C}$ $\forall f \in \mathrm{Hom}(A,B)$, the diag comm.

$$\begin{tikzcd} \mathcal{F}A \arrow[r, "\eta_A"] \arrow[d, "\mathcal{F}(f)"'] & \mathcal{G}A \arrow[d, "\mathcal{G}(f)"] \\ \mathcal{F}B \arrow[r, "\eta_B"] & \mathcal{G}(B) \end{tikzcd}$$

Ex $\mathcal{F}$ = identity functor on Fd-vec/k
$\mathcal{G} = D^2$ " Fd-vec/k

$$\eta_V : V \to V^{**}$$

Since each η_V is an $\cong$,
say it is a natural isomorphism of functors