Math 620 - Final Exam: Due Monday May 4, 2009

Instructions: Do all the problems. You may use only your class notes and the textbook, no other sources or collaboration.

1. Let $f(x) \in \mathbb{Q}[x]$ have degree n and let K be a splitting field of $f(x)$ over $\mathbb{Q}$. Suppose that the Galois group $\mathrm{Gal}(K/\mathbb{Q})$ is isomorphic to the symmetric group S_n.

a. Show that $f(x)$ is irreducible over $\mathbb{Q}$.

b. If $n > 2$ and α is a root of $f(x)$ in K, show that the only automorphism of $\mathbb{Q}(\alpha)$ is the identity.

c. If $n \geq 4$, show that $\alpha^n \notin \mathbb{Q}$.

2. Let A be a 9×9 matrix with complex entries, characteristic polynomial $(x^2+1)^3(x+2)^3$ and minimal polynomial $(x^2+1)^2(x+2)$.

a. Find the trace and determinant of A.

b. How many different conjugacy classes of matrices with this property are there in $GL_n(\mathbb{C})$? How many in $GL_n(\mathbb{Q})$?

c. Write down such a matrix with rational entries.

3. Page 877 $\#6, 9$.

4. Let $G = GL_n(\mathbb{C})$ and let $B \leq G$ be the Borel subgroup of invertible upper triangular matrices. Show that Maschke's theorem does not hold for B, i.e. demonstrate a representation that is not completely reducible.

5. (Converse to Maschke's Theorem) Suppose G is a finite group and $p \mid |G|$. Let F be a field of characteristic p. Let $\Delta(G)$ be the kernel of the augmentation map $\epsilon : FG \to F$ defined by:

$$\epsilon(\sum a_g g) = \sum a_g.$$

a. Check that $\Delta(G)$ (called the *augmentation ideal*) is a submodule of FG and that $FG/\Delta(G) \cong F$, the trivial representation.

b. Let $\sigma = \sum_{g \in G} g$. Check that σ spans a one-dimensional trivial submodule of FG and that $\sigma \in \Delta(G)$.

c. Observe that we have a series of submodules $0 \subset \langle \sigma \rangle \subseteq \Delta(G) \subset FG$ which, when refined to a composition series, has at least two copies of the trivial module in it.

d. Show that FG is not a semisimple algebra.