Math 620 - Midterm Exam #2- April 7, 2009

Instructions: Choose four of the six problems.

1. Give an example of:

   a. An inseparable field extension.

   b. A polynomial in $\mathbb{Q}[x]$ with Galois group cyclic of order 4.

   c. A ring that is not Noetherian.

   d. A local ring.

   e. A polynomial in $\mathbb{Q}[x]$ with Galois group isomorphic to D_8, the dihedral group of eight elements.

   f. Determine the minimal polynomial for $\sqrt{2}+\sqrt{5}$ over $\mathbb{Q}$.

2. Let $V = \mathcal{Z}(xy - z) \in \mathbb{A}^3$. Prove that V is isomorphic to $\mathbb{A}^2$ by providing an explicit isomorphism $\phi : \mathbb{A}^2 \to V$ and associated k-algebra isomorphism $\tilde{\phi} : k[V] \to k[\mathbb{A}^2]$, along with their inverses.

3. Determine the splitting field and Galois group of $x^3 - 2$ over $\mathbb{Q}$ and for each subgroup compute the corresponding fixed field.

4. Let L be the Galois closure of the finite extension $\mathbb{Q}(\alpha)$ of $\mathbb{Q}$. For any prime dividing the degree $[L : \mathbb{Q}]$, prove that there is a subfield F of L with $[L : F] = p$ and $L = F(\alpha)$.

5. Prove that an *irreducible* affine variety is connected in the Zariski topology.

6. Let k be algebraically closed and let $V_1 \supset V_2 \supset \cdots$ be a descending chain of varieties in $\mathbb{A}^n \cong k^n$. Prove that $V_m = V_{m+1} = \cdots$ for some m.