Math 620 Spring 2009- Midterm Exam #1

Instructions: Choose five of the seven problems.

1. Consider the group $G = GL_4(F_3)$ where F_3 denotes the field with 3 elements. How many conjugacy classes of elements of order two does G have? Give a representative from each class.

2. Let A be an $n \times n$ matrix over an algebraically closed field F and let $F[A]$ denote the linear span of the matrices $I = A^0, A, A^2, A^3, \ldots$, so $F[A]$ is an F-algebra. Prove that A is diagonalizable if and only if $F[A]$ contains no nonzero nilpotent element.

Hint: If A is diagonalizable, show that we can assume it is diagonal WLOG. For the converse, consider the factorization of the minimal polynomial $m_A(x)$ of A. If $m_A(x)$ has a repeated root, construct a nonzero nilpotent element.

3. Define what it means for two functors to be naturally isomorphic, and give an example.

4. Determine all possible rational canonical forms for linear transformations with characteristic polynomial $x^2(x^2+1)^2$.

5. Let V be a 3-dimensional vector space and let $T : V \to V$ be the linear map given by the matrix

$$\begin{pmatrix} 1 & -1 & 2 \\ 1 & 0 & 1 \\ -2 & 3 & 1 \end{pmatrix}$$

in terms of a basis $\{v_1, v_2, v_3\}$.

a. Give a basis for the vector space $\Lambda^2(V)$.

b. Write the matrix for $\Lambda^2(T) : \Lambda^2(V) \to \Lambda^2(V)$ in this basis.

c. Repeat for $\Lambda^3(V)$.

6. List the isomorphism classes of finite abelian groups of order 200.

7. Prove that any $n \times n$ matrix A with complex entries and with the property that $A^3 = A$ is diagonalizable. Is this property true over an arbitrary field? Either prove it or give a counterexample.