Math 619 Midterm Exam #1 SOLUTIONS

1. a. A group action on A is faithful if $g \cdot a = a \ \forall a \in A$ implies $g = e$.

b. $(D_8)' = \{e, r^2\}, \quad (S_4)' = A_4$.

c. Suppose $H, K \leq G$ with $H \leq N_G(K)$. Then $HK \leq G$, $K \trianglelefteq HK$, $H \cap K \trianglelefteq H$ and $HK/K \cong H/H \cap K$.

d. $\{e\}, \{r^2\}, \{r, r^3\}, \{s, sr^2\}, \{sr, sr^3\}$

e. $Z(G) = \{z \in G \mid zg = gz \ \forall g \in G\}$.

2. a. Easy check shows that $i_g \circ i_{g^{-1}}(x) = x = i_{g^{-1}} \circ i_g(x)$ so i_g is a bijection. Also:

$$i_g(xy) = gxyg^{-1} = gxg^{-1}gyg^{-1} = i_g(x)i_g(y)$$

so i_g is an automorphism of G.

b. Let $g, h \in G$. Then

$$i_{gh}(x) = ghx(gh)^{-1} = ghxh^{-1}g^{-1} = i_g(hxh^{-1}) = i_g \circ i_h(x)$$

so $i_{gh} = i_g \circ i_h$ and the map i is a homomorphism $G \to \mathrm{Aut}(G)$..

c. The map i_g is trivial when $gxg^{-1} = x \ \forall x \in G$, so the kernel of i is $Z(G)$.

d. Let $\psi \in \mathrm{Aut}(G)$ and $g \in G$. We must show $\psi \circ i_g \circ \psi^{-1}$ is in the image of i. But:

$$\begin{aligned} \psi \circ i_g \circ \psi^{-1}(x) &= \psi(g\psi^{-1}(x)g^{-1}) \\ &= \psi(g)\psi(\psi^{-1}(x))\psi(g)^{-1} \\ &= \psi(g)x\psi(g)^{-1} \\ &= i_{\psi(g)}(x). \end{aligned}$$

3. We have

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$$

To get an isomorphism we can map $(1,2)$ to any of the matrices of order 2, for example to $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and map the three cycle (123) to one of the matrices of order 3.

4. Done in class.

5. The kernel contains $\{e\}$ so is nonempty. Let $x, y \in \ker f$ and $g \in G$. Then $f(xy^{-1}) = f(x)f(y)^{-1} = e$ so $xy^{-1} \in \ker f$ and it is a subgroup. Finally $f(gxg^{-1}) = f(g)f(x)f(g)^{-1} =$

$f(g)ef(g)^{-1} = e$ so $gxg^{-1} \in \ker f$ and it is normal.

Let $x, y \in f^{-1}(B)$ so $f(x) = b_1 \in B$ and $f(y) = b_2 \in B$. Notice $f(e) = e \in B$ so the set is nonempty. Next:

$$f(xy^{-1}) = f(x)f(y)^{-1} = b_1 b_2^{-1} \in B$$

since $B \leq H$. Thus $xy^{-1} \in f^{-1}(B)$ so $f^{-1}(B) \leq G$.

6. Choose $x = (12)(34)$ and $y = (123)$ for example to find this presentation:

$$A_4 = \langle x, y \mid x^2 = y^3 = (xy)^3 = e \rangle$$