Math 619 Midterm Exam #2- November 1, 2012

1. Short Answer- no work need be shown. (30 points)

a. Give generators for a Sylow 3-subgroup of S_9.

b. Define a *nilpotent group* and *nilpotence class.*

c. Give an example of a group that is solvable but not nilpotent.

d. Give an example of a group G with a normal subgroup $H \trianglelefteq G$ which does not have a complement.

e. List the isomorphism classes of abelian groups of order 36.

2. (10 points) Suppose $H \trianglelefteq G$ and K is a characteristic subgroup of H. Prove $K \trianglelefteq G$.

3. (10 points) Prove that D_{8n} is not isomorphic to $Z_2 \times D_{4n}$.

4. (15 points) Let P be a Sylow p-subgroup of G and let $N = N_G(P)$. Prove that $N_G(N) = N$.

5. (15 points) Prove that there are no simple groups of order $336 = 2^4 \cdot 3 \cdot 7$. Hint: Look at the permutation action of G on $\mathrm{Syl}_7(G)$.

6. (20 points) Classify the isomorphism types of groups of order 18, and give a presentation for each.

You may find it useful to know the group $GL_2(3)$ has order $48 = (3^2 - 1)(3^2 - 3)$ and every element of order 2 is conjugate to either

$$\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \text{ or } \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}.$$

1a. $\{(1,2,3),(1,4,7)(2,5,8)(3,6,9)\}$

b. Define the upper central series by $Z^i(G)$ by $Z^0(G) = \{e\}$ and $Z^i(G)/Z^{i-1}(G) = Z(G/Z^{i-1}(G))$. Then G is nilpotent of class c if $Z^{c-1}(G) < Z^i(G) = G$.

c. S_4 is solvable but not nilpotent.

d. Let $G = Q_8$ and $H = Z(Q_8) = \{\pm 1\}$.

e. $Z_{36}, Z_{18} \times Z_2, Z_{12} \times Z_3, Z_6 \times Z_6$

2. Let $g \in G$ and $i_g(x) = gxg^{-1}$ be the corresponding inner automorphism of G. Since $H \trianglelefteq G$ then i_g restricts to an automorphism of H, and hence $i_g(K) = gKg^{-1} = K$ as K is characteristic in H. Thus $K \trianglelefteq G$.

3. Recall that D_{2n} has trivial center if $n > 1$ is odd and center $\{e, r^{n/2}\}$ if n is even. This is easy to show. The reflections sr^i do not commute with r since $rsr^ir^{-1} = sr^{i-2}$. Moreover $sr^is^{-1} = r^{-i}$ so powers of r do not commute with s unless n is even and $i = n/2$. Turning to the problem then we see D_{8n} has a center of order 2 but $Z_2 \times D_{4n}$ has a center of order 4, namely $Z_2 \times \{e, r^{2n}$, so the groups are not isomorphic.

4. Let $x \in N_G(N)$. Then $xNx^{-1} = N$ so $xPx^{-1} \leq N$. Thus xPx^{-1} is a Sylow p-subgroup of N but $P \trianglelefteq N$ is a normal Sylow p-subgroup. This means N has a unique Sylow p-subgroup so $xPx^{-1} = P$, i.e. $x \in N$. Thus $N_G(N) \leq N$. The other containment is automatic, so $N = N_G(N)$.

5. Suppose G is simple of order 336. By Sylow's theorem $n_7 = 8$. Choose $P_7 \in \mathrm{Syl}_7(G)$ and let $N = N_G(P_7)$. Then $n_7 = [G : N] = 8$ so $|N| = 42$.

Action on left cosets of N gives a homomorphism $\rho : G \to S_8$, which is necessarily injective as G is simple. So assume WLOG that $G \leq S_8$. Recall that if $G \not\leq A_8$ then $G \cap A_8$ is a normal subgroup of index 2, so we may assume further $G \leq A_8$ WLOG.

Elements of order 7 in S_8 are 7-cycles, so WLOG let $P_7 = \langle(1,2,3,4,5,6,7)\rangle$. Then:

$$N_{S_8}(P_7) = \langle(1,2,3,4,5,6,7),(1,4,6,5,2,7)\rangle$$

has order 42. However $(1,4,6,5,2,7)$ is odd so not in A_8. Thus

$$|N_{A_8}(P_7)| = 42/2 = 21.$$

But $N_G(P_7) \leq N_{A_8}(P_7)$ and $42 \not\leq 21$, so we have a contradiction. The normalizer of a Sylow 7 in G is twice the size of the normalizer in A_8.

6. By Sylow's theorem $n_9 = 1$ so we have $P_9 \trianglelefteq G$. Let Z_2 be a Sylow 2-subgroup. Then $P_9Z_2 = G$ and $P_9 \cap Z_2 = \{e\}$ so $G \cong P_9 \rtimes Z_2$.

Suppose first that P_9 is cyclic, isomorphic to Z_9. Then $\mathrm{Aut}\, Z_9$ is abelian and has order $\phi(9) = 6$. In particular it has a unique automorphism of order 2 that inverts the generator. So we have one nontrivial map $Z_2 \to \mathrm{Aut}\, Z_9$ and the trivial one, giving two groups:

$$G_1 \cong Z_9 \times Z_2 \cong Z_{18} = \{x \mid x^{18} = 1\}$$
$$G_2 = \{x, y \mid x^9 = y^2 = 1, yxy^{-1} = x^{-1}\}$$

Next suppose $P_9 \cong Z_3 \times Z_3$. Since $\mathrm{Aut}(Z_3 \times Z_3) \cong GL_2(3)$, we are looking for maps $\phi : Z_2 \to GL_2(3)$. We know from exercise 6 that maps with images conjugate subgroups give the same semidirect product, so we get at most two nonabelian groups together with the direct product. The two matrices give us the actions:

$$G_3 = \{a, b, y \mid a^3 = b^3 = y^2 = 1, ab = ba, yay^{-1} = a, yby^{-1} = b\} \cong Z_3 \times Z_3 \times Z_2.$$

$$G_4 = \{a, b, y \mid a^3 = b^3 = y^2 = 1, ab = ba, yay^{-1} = a^{-1}, yby^{-1} = b\}$$

$$G_5 = \{a, b, y \mid a^3 = b^3 = y^2 = 1, ab = ba, yay^{-1} = a^{-1}, yby^{-1} = b^{-1}x\}$$

To see the last two groups are not isomorphic one can compute that G_5 has trivial center while $b \in Z(G_4)$.