Math 619 Final Exam SOLUTIONS - December 11, 2012

1. Short Answer- no work need be shown. (40 points)

a. $\mathbb{Z}[x]$

b. -4+10i

c. $R = F[x_1, x_2, \dots,]$ is a finitely generated left R module but the ideal $(x_1, x_2, \dots)$ is not finitely generated.

d. All are of the form $M_c = \{f \mid f(c) = 0\}$ where $0 \leq c \leq 1$.

e. Let P be a prime ideal in an integral domain R and let $f(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_0 \in R[x]$. Suppose all the $a_i \in P$ and $a_0 \notin P^2$. Then $f(x)$ is irreducible in $R[x]$.

f. $M_2(F)$. The set of nilpotent matrices is not even closed under addition.

g. 48

h. $\{e, (12), (123), (1234), (12)(34)\}$

2. (20 points) Claim: $\eta(R)$ is the intersection of every prime ideal in R.

Proof: Clearly nilpotent elements are in every prime ideal, so one containment is clear. To show equality we must prove that for a not nilpotent, there exists a prime ideal not containing a. So suppose a is not nilpotent. Set $\mathcal{S}$ be the set of ideals that do not contain any power a^n, so by assumption $\{0\} \subseteq \mathcal{S}$. Clearly Zorn's lemma applies and $\mathcal{S}$ has a maximal element P. Check that P is prime

b implies c

Suppose $x + \eta(R) \neq 0$. Then $x \notin \eta(R)$ so x is a unit. Thus $x^{-1} + \eta(R)$ is the inverse of $x + \eta(R)$ so $R/\eta(R)$ is a field.

c implies a

If $R/\eta(R)$ is a field then $\eta(R)$ is maximal and hence prime. Any other prime ideal contains $\eta(R)$ by the above, so there cannot be any as $\eta(R)$ is maximal. Thus $\eta(R)$ is the unique prime ideal.

a implies b

Suppose x is neither nilpotent nor a unit. Since x is not a unit then (x) is proper and thus lies in some maximal ideal $\mathfrak{m}$ whichby (a) must be the unique prime ideal, by the lemma this is $\eta(R)$, contradicting x not being nilpotent.

3. (10 points) Suppose M is cyclic generated by x. Then $\{rx \mid r \in R\} = M$. Since $M/N = \{m + N \mid m \in M\}$ it is clear that M/N is cyclic, generated by $x + N$.

4. (10 points) $\mathbb{Z}[i]/(7)$ is a field of 49 elements as we saw on the homework since 7 is congruent to 3 mod 4. Alternately once can take $\mathbb{Z}/7\mathbb{Z}[x]/(p(x))$ where $p(x)$ is an irreducible quadratic.

5. (20 points) Suppose $m \in \operatorname{Ker}\psi \cap \operatorname{Im}\psi$. Then $\psi(m) = 0$ and there is a $u \in M$ so $m = \psi(u)$. Thus $m = \psi(\psi(u)) = 0$, so the two spaces intersect trivially. They are both submodules of M. Finally note that for $m \in M$ we have

$$m = \psi(m) + (m - \psi(m)) \in \operatorname{Im}\psi + \operatorname{Ker}\psi$$

so the criterion for an internal direct sum is verified.

6. (10 points) The polynomial factors as $(x+2)(x+1)$ so it is is reducible in $\mathbb{Z}[x]$. However in the power series ring $1 + x$ is a unit so this is not a nontrivial factorization. Recall that:

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 \cdots .$$

7. (20 points) Let P be a prime ideal in a commutative ring R with identity.

a. This is just the definition of prime ideal, if $a, b \notin P$ then $ab \notin P$.

b. Notice that the units in R_P are just fractions of the form $\frac{x}{y}$ where neither x nor y is in P. Observe that the set $PR_P = \{\frac{p}{y} \mid p \in P\}$ is an ideal of R_P and it contains every noninvertible element. This immediately implies it is maximal. Since any proper ideal cannot contain any units, this ideal contains every proper ideal, and is thus the unique maximal ideal.

c. Notice that $\mathbb{Z}_{(2)}$ is just rational numbers with odd denominators. Now suppose a is also odd. Then $\frac{a}{b} = \frac{b}{b} + \frac{a-b}{b}$ and $a - b$ is even so $\frac{a-b}{b} \in PR_P$ and we have:

$$\frac{a}{b} = \overline{1}.$$

When a is even then $\frac{a}{b} = \overline{0}$. Thus the quotient field has just two elements, $\overline{0}$ and $\overline{1}$.

8. (20 points) F^* is a finite abelian group, suppose it has n elements. Write it in invariant factor form:

$$C_{n_1} \times \cdots \times C_{n_s}$$

where $n_i \mid n_{i-1}$.

We have $n = n_1 n_2 \cdots n_2$. Notice that every element in this group has order dividing n_1, and is thus a root of the polynomial $x^{n_1} - 1$. However $F[x]$ is a UFD, so a polynomial of degree n_1 has at most n roots. Thus $n \leq n_1 \leq n$ so $n = n_1$ and the group is cyclic.

9. (20 points) Let $I \subseteq R$ be an ideal in a ring R with identity, and let M be a nonzero R-module.

a. $IM = \{i_1 m_1 + \cdots + i_s m_s \mid i_t \in I, m_t \in M\}$. It is clearly closed under addition and since $ri_t \in I$, it is also closed under the left action of R, so it is a submodule.

b. Suppose $I^k = 0$. If $IM = M$ then $0 = I^k M = IM = M$, a contradiction. Thus IM is proper. If M is simple though then 0 is the only proper submodule, so necessarily $IM = 0$.
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c. Let I be the augmentation ideal of FP, the kernel of the augmentation map $\epsilon : FP \to F$. We proved on the homework that this ideal is nilpotent and codimension 1. Let S be a simple FP-module. Every thing in FP is of the form $\lambda \cdot 1 + i$ for $i \in I$,$\lambda \in F$. Now let $0 \neq s \in S$. Then since S is simple we have:

$$S = FPs = \{(\lambda \cdot 1 + i)s\} = \{\lambda s\}.$$

This shows that S is one-dimensional and that $\kappa \in FP$ acts on this one-dimensional space by $\epsilon(\kappa)$ where ϵ is the augmentatoin map (since $\epsilon(\lambda \cdot 1 + i) = \lambda$.) That is, we have shown that $S \cong FP/I$.

10. (10 points) If $[G : H] = p$ we get a homomorphism $f : G \to S_p$ via the action on the cosets. For $g \notin H$ we have $gH \neq H$ so $\ker f \leq H < G$ and $G/\ker f \cong \operatorname{Im} f \leq S_p$. If $\ker f < H$ this gives a contradiction, as $[H : \ker f]$ would be divisible by only primes $\geq p$ but $p!$ is divisible by one power of p together with smaller primes. Thus $H = \ker f$ so $H \trianglelefteq G$.

11. (20 points) First observe conjugation preserves the set of maximal subgroups so $\Phi(G) \trianglelefteq G$ (actually it is characteristic). Let P be a Sylow subgroup of $\Phi(G)$. By the Frattini argument we have $G = N_G(P)\Phi(G)$. If $N_G(P)$ were proper, it would lie in some maximal subgroup M. Of course $\Phi(G)$ would also lie in M meaning $N_G(P)\Phi(G) \leq M < G$, a contradiction. Thus $N_G(P) = G$ and P is normal in G, in particular it is normal in $\Phi(G)$. Since all Sylows are normal, we have that $\Phi(G)$ is nilpotent.