Math 619 Final Exam - December 11, 2012

1. Short Answer- no work need be shown. (40 points)

a. Give an example of a UFD that is not a PID.

b. Find a generator for the ideal $(-4+10i, 58)$ in $\mathbb{Z}[i]$.

c. Give an example of a ring which is finitely generated with an ideal which is not finite generated.

d. Describe the maximal ideals in $C[0,1]$.

e. State the Eisenstein criterion.

f. Illustrate an example of a ring where the set of nilpotent elements does not form an ideal.

g. How many elements of order 7 are there in a simple group of order 168?

h. Give a set of conjugacy class representatives in the symmetric group S_4.

2. (20 points) Let R be a commutative ring with identity. Recall $\eta(R)$ is the nilradical of R, the set of all nilpotent elements. Prove the following are equivalent:

a. R has exactly one prime ideal.

b. Every element of R is either nilpotent or a unit.

c. $R/\eta(R)$ is a field.

3. (10 points) Let M be a cyclic R module. Prove that any quotient module M/N is also cyclic.

4. (10 points) Construct a field with 49 elements.

5. (20 points) Let $\psi : M \to M$ be an R-module homomorphism such that $\psi \circ \psi = \psi$. Prove that

$$M \cong \operatorname{Ker}\psi \oplus \operatorname{Im}\psi$$

as R-modules.

6. (10 points) Recall that $\mathbb{Z}[[x]]$ is the ring of formal power series with integer coefficients. Prove that $x^2 + 3x + 2$ is irreducible in $\mathbb{Z}[[x]]$ but is reducible in $\mathbb{Z}[x]$.

7. (20 points) Let P be a prime ideal in a commutative ring R with identity.

a. Show the set $D := R - P$ is multiplicatively closed.

b. Let R_P denote the ring of fractions $D^{-1}R$ with respect to this multiplicatively closed set. Show R_P has a unique maximal ideal $\mathfrak{m}$.

c. Describe the field $R_P/\mathfrak{m}$ for $R = \mathbb{Z}$, $P = (2)$ and $\mathfrak{m}$ the unique maximal ideal.

8. (20 points) Let F be a finite field and F^* the multiplicative group of nonzero elements. Prove that F^* is a cyclic group. Hint: Use the classification of finite abelian groups and properties of $F[x]$.

9. (20 points) Let $I \subseteq R$ be an ideal in a ring R with identity, and let M be a nonzero R-module.

a. Define IM and prove it is a submodule of M.

b. Suppose further that I is a nilpotent ideal. Prove that IM is a proper submodule, and conclude that I annihilates any simple R module.

c. Let F be a field of characteristic p and P a finite p-group. Prove that FP has a unique simple module, which is one-dimensional.

10. (10 points) Let p be the smallest prime dividing the order of a group G. Prove that any subgroup of index p must be normal.

11. (20 points) Let G be finite and recall the Frattini subgroup $\Phi(G)$ is the intersection of all maximal subgroup of G. Prove that $\Phi(G)$ is nilpotent.