Math 619 Midterm Exam #2- November 14, 2011

1. Short Answer- no work need be shown. (40 points)

a. Give the invariant factor decomposition for the isomorphism classes of abelian groups of order 72.

$$C_{72},\ C_{36} \times C_2,\ C_{18} \times C_2 \times C_2,\ C_{24} \times C_3,\ C_{12} \times C_6,\ C_6 \times C_6 \times C_2.$$

b. Let $H \trianglelefteq G$. Define a *complement* for the subgroup H. Give an example where H does not have a complement.

A complement is a subgroup $K \leq G$ such that $G = HK$ and $H \cap K = 1$. For $G = Q_8$ and $H =< i >$ there is no complement.

c. Give an example of a unique factorization domain which is not a principal ideal domain.

$\mathbb{Z}[x]$

d. Give an example of a group which is solvable but not nilpotent.

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e. Give an example of a maximal ideal in $C[0, 1]$, the ring of continuous real-valued functions on $[0, 1]$.

The maximal ideals are all of the form $M_c := \{f \mid f(c) = 0\}$ where $c \in [0, 1]$.

f. Let I, J be ideals in a ring R. Describe the ideal IJ.

$$IJ = \{i_1 j_1 + i_2 j_2 + \cdots + i_s j_s \mid i_t \in I, j_t \in J, s \geq 1.\}$$

g. Define an *integral domain*.

A commutative ring with identity and no zero divisors.

h. Let G be a group. Define the *Frattini subgroup* of G.

The intersection of all maximal subgroups, if there are any. Otherwise it is defined as G.

2. (15 points) State the Chinese Remainder Theorem.

Let R be a commutative ring with identity and $I_1, I_2, \dots, I_s$ be ideals which are pairwise comaximal. Then $I_1 I_2 \cdots I_s = I_1 \cap I_2 \cdots \cap I_s$ and:

$$R/I_1 I_2 \cdots I_s \cong R/I_1 \times R/I_2 \times \cdots \times R/I_s.$$

3. (25 points) Classify the isomorphism types of groups of order 75.

There are three isomorphism types, two abelian and one nonabelian. By Sylow's theorem the Sylow 5 subgroup P_5 has order 25 and is normal. Any Sylow 3 subgroup will be a complement so G must be a semidirect product $P_5 \rtimes P_3$. Since it has order 5^2, we have $P_5 \cong C_{25}$ or $C_5 \times C_5$. The former has an automorphism group of order $\phi(25) = 20$, which has no elements of order 3. Thus this example is just:

$$G_1 \cong C_{25} \times C_3 \cong C_{75}.$$

When $P_5 \cong C_5 \times C_5$ we get the direct product:

$$G_2 \cong C_5 \times C_5 \times C_3 \cong C_{15} \times C_5.$$

However $\mathrm{Aut}(C_5 \times C_5) \cong \mathrm{GL}_2(5)$ has order $(5^2-1)(5^2-5) = 2^5 \cdot 3 \cdot 5$ so there is a nontrivial semidirect product also.

The Sylow 3 subgroups of $\mathrm{GL}_2(5)$ are cyclic of order 3 and are all conjugate. Thus any nontrivial map $\phi : C_3 \to \mathrm{GL}_2(5)$ will give the same semidirect product (by the famous "exercise 6"):

$$G_3 \cong (C_5 \times C_5) \rtimes C_3.$$

Finding an explicit matrix of order 3 allows one to write down a presentation. For example $\begin{pmatrix} 0 & 2 \\ 2 & -1 \end{pmatrix}$ yields:

$$G_3 \cong \langle x, y, z \mid x^5 = y^5 = z^3 = 1, xy = yx, zxz^{-1} = y^2, zyz^{-1} = x^2y^{-1} \rangle.$$

4. (20 points) Suppose R is a commutative ring with identity and with the property that every ideal is finitely generated. Suppose

$$0 \subset I_0 \subseteq I_1 \subseteq I_2 \subseteq \cdots R$$

is an ascending chain of ideals. Prove the chain terminates, i.e. that there exists some $s \geq 0$ such that

$$I_s = I_{s+1} = I_{s+2} = \cdots$$

Let $I = \cup_{i=j}^{\infty} I_j$ which is easily seen to be an ideal, so is finitely generated, let $I = (r_1, r_2, \dots, r_m)$. Each r_i is in the union, so is in some $I_{t(i)}$. Choosing s as the the maximum $t(i)$ we find the generators all lie in some I_s and thus

$$I = I_s = I_{s+1} = \cdots$$

as desired.