Math 619 Midterm Exam #1 SOLUTIONS- October 7, 2011

1. Short Answer- no work need be shown. (40 points)

a. Let G act on A and $a \in A$. Define the *orbit* of a and the *stabilizer* of a.

ANSWER: The orbit of a is the set $\Theta(a) = \{g \cdot a \mid g \in G.\}$. The stabilizer is the subgroup $G_a = \{g \in G \mid g \cdot a = a\}$.

b. Let G be a finite group. Define the *composition factors* of G.

ANSWER: Choose a series

$$e = G_0 \trianglelefteq G_1 \cdots \trianglelefteq G_s = G$$

with each G_i/G_{i-1} simple. Then the composition factors are the simple groups $\{G_i/G_{i-1}\}$. The Jordan-Holder theorem guarantees this set is independent of the choice of composition series.

c. What is the automorphism group of the Klein 4-group? Of the symmetric group S_4?

ANSWER: $\text{Aut}(V) \cong S_3$ and $\text{Aut}(S_4) \cong S_4$.

d. What is the center of the dihedral group D_8?

ANSWER: $\{e, r^2\}$.

e. Give a set of conjugacy class representatives for the symmetric group S_4. Repeat for the alternating group A_4.

ANSWER: For S_4: $\{e, (12), (123), (1234), (12)(34)\}$ and for A_4: $\{e, (12)(34), (123), (132)\}$.

f. Define the *commutator subgroup* G' of a group G.

ANSWER: It is the subgroup generated by the set $\{xyx^{-1}y^{-1} \mid x, y \in G.\}$.

2. (20 points) Let M and N be normal subgroups of G such that $G = MN$. Prove that $G/(M \cap N) \cong G/M \times G/N$.

ANSWER: Define a homomorphism $\phi : G \to G/M \times G/N$ by $\phi(g) = (gM, gN)$. Then ϕ is clearly a homomorphism and $(gM, gN) = (M, N)$ if and only if $g \in M \cap N$, so $\text{Ker}\,\phi = M \cap N$.

The result is immediate by the first isomorphism theorem if we can show ϕ is onto. Choose $g_1, g_2 \in G$. Since $G = MN$ is a subgroup then $MN = NM$ and we can write $g_1 = m_1n_1$ and $g_2 = n_2m_2$. Then $g_1N = m_1N$ so $\phi(m_1) = (M, g_1N)$. Similarly $g_2M = n_2M$ so $\phi(n_2) = (g_2M, N)$. Thus

$$\phi(m_1n_2) = (g_2M, g_1N)$$

so ϕ is onto.

3. (20 points) Let G be an infinite simple group. Prove that G does not have a subgroup of finite index $n > 1$.

ANSWER: Suppose $H \leq G$ with $[G : H] = n$. The action of G on the left cosets of H gives a homomorphism $\phi : G \to S_n$ which is nontrivial. The kernel of this homomorphism is a proper normal subgroup of index $\leq n!$, since

$$G/\operatorname{Ker}\phi \cong \phi(G) \leq S_n.$$

This contradicts G being simple.

4. (20 points) Let $p < q$ be primes. Prove that every finite group of order p^2q has a normal Sylow subgroup.

ANSWER: Suppose the Sylow q subgroup is not normal. By Sylow's theorem, $n_q = 1+tq$ for $t > 0$ and n_q divides p^2. If $1 + tq = p$ this would contradict $p < q$. Thus $1 + tq = p^2$ so $tq = (p-1)(p+1)$. Because $q > p$, this forces $t = 1$ and $q = p+1$.

This only happens when $p = 2, q = 3$. In this case though, if $n_3 = 4$ we get a homomorphism ϕ from G into S_4 from the action on the four Sylow 3 subgroups. By counting we see G has eight elements of order 3, and we immediately see that ϕ is 1-1 with image A_4. Thus $G \cong A_4$, which has a normal Sylow 2-subgroup.