Math 619 Final Exam - December 14, 2011

1. Short Answer- no work need be shown. (40 points)

a. The set of upper triangular matrices with 1's on the diagonal.

b. The center of A_4 is trivial so the ascending central series is just $1 \subseteq 1 \cdots$. However $[A_4, A_4] = V$ the Klein 4-group and $[A_4, V] = V$ so we have $A_4 \supset V \supseteq V \cdots$.

c. $2 \cdot 6!$. Remember for $n = 6$ only we have $|\operatorname{Out}(S_6)| = 2$.

d. A_4.

e. $(123)(45678)(9, 10, 11, 12, 13, 14, 15)$ has order 105.

f. A complement is a subgroup $K \leq G$ with $H \cap K = 1$ and $HK = G$. The subgroup $Z(Q_8) \trianglelefteq Q_8$ does not have a complement.

g. (x) is prime but not maximal in $\mathbb{Z}[x]$.

h. $M_2(2)$, the ring of 2×2 matrices over a field of two elements has 6 invertible elements.

i. On p. 69

j. $H = \{e, r\} \trianglelefteq N = \{e, r, s, sr\} \trianglelefteq D_8$ but H is not normal in D_8.

2. (15 points) Recall that $\mathbb{Q} \otimes_{\mathbb{Z}} A = 0$ for any torsion (in particular finite) abelian group. So it's clear that $\Pi_{i=1}^{\infty}(\mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Z}/2^i\mathbb{Z}) = 0$ since each term is zero.

To show that $\mathbb{Q} \otimes_{\mathbb{Z}} \Pi_{i=1}^{\infty}\mathbb{Z}/2^i\mathbb{Z}$ is nonzero actually requires a little more that we've done in class. You can show that the element $1 \otimes (1, 1, 1, \dots)$ generates a nonzero subgroup isomorphic to $\mathbb{Z}$. For a complete proof we need to show that $\mathbb{Q}$ is a *flat* $\mathbb{Z}$ module.

3. (15 points) Let $n \in N$. Since $\overline{\phi}$ is surjective there is an $m_1 \in M$ and $in_1 \in IN$ so that $\phi(m_1) = n + in_1$ (Let's abuse notation and write in_1 for an element of IN even though we know it's a sum of such elements). So $n = \phi(m_1) - in$. Now repeat for $n_1 = m_2 + in_2$ to we see that:

$$n = \phi(m_1) - \phi(im_2) - i^2\phi(n_2)$$

again abusing notation. Repeating k times where $I^k = 0$ we find that $n \in \phi(M)$ so ϕ is onto.

4. (15 points) Let M be irreducible and $0 \neq m \in M$. Since R is free we have a module homomorphism $\phi : R \to M$ given by $\phi(r) = rm$. The image is a nonzero submodule so ϕ is onto with kernel I, so $M \cong R/I$. However any ideal $I \subset JsubsetR$ would be a nontrivial submodule of R/I. Since M is irreducible then we have I is maximal. Conversely if I is maximal then R/I is a left R module which is irreducible by the same idea.

5. (15 points) As $F[x]$ is a UFD we can factor $p(x) = f_1(x)^{m_1} \cdots f_s(x)^{m_s}$ into distinct irreducibles $f_i(x)$. The correspondence theorem says we are looking for ideals of $F[x]$ that contain $p(x)$. Since $F[x]$ is a PID we are looking for polynomials $q(x)$ so that $q(x) \mid p(x)$. Thus the ideals are exactly $(q(x))/(p(x))$ where:

$$q(x) f_1(x)^{n_1} \cdots f_s(x)^{n_s}, 0 \leq n_i \leq m_i.$$

6. (10 points) If $x, y \in \ker \phi$ then $\phi(xy^{-1}) = ee = e$ so it is a subgroup. Also for $g \in G$ we have $\phi(gxg^{-1}) = \phi(g)\phi(x)^{-1}\phi(g)^{-1} = \phi(e) = e$ so it is a normal subgroup. If $e \neq x \in \ker \phi$ then $\phi(x) = \phi(e) = e$ so ϕ is not 1-1. Moreover if $x \neq y$ but $\phi(x) = \phi(y)$ then $\phi(xy^{-1}) = 1$ so $\ker \phi \neq e$.

7. (15 points) Suppose $\phi^{-1}(P) \neq R$ and choose x, y with $xy \in \phi^{-1}(P)$. Then $\phi(xy) = \phi(x)\phi(y) \in P$ so either $\phi(x)$ or $\phi(y)$ is in P, so either x or y is in $\phi^{-1}(P)$, so it is prime. Moreover $\phi(rx) = \phi(r)\phi(x) \in P$ since P is an ideal, so $rx \in \phi^{-1}(P)$, which shows it is an ideal.

8. (10 points) The field can be $(\mathbb{Z}/3\mathbb{Z})[x]/(x^2+1)$. The nine elements are $a + bx$ with $a, b \in \mathbb{Z}/3\mathbb{Z}$ and multiplication table obtained from the rule $x^2 = 2$.

9. (15 points) Let $n \in N$ and $g \in G$. Consider the element $gng^{-1}n^{-1}$. It lies in N since N is normal so $gng^{-1} \in N$. It's also a commutator so it's in G'. Thus it is trivial, i.e. $gn = ng$ so $N \subseteq Z(G)$.

10. (10 points) By Gauss Lemma it is enough to show $p(x)$ is irreducible over $\mathbb{Z}$. The rational root test says the possible roots over $\mathbb{Q}$ are ± 2, ± 1 which are not roots. So any factorization is into quadratics. So suppose:

$$x^4 + 5x^2 + 3x + 2 = (x^2 + ax + b)(x^2 + cx + d).$$

We obtain:

$$a + c = 0. \; b + ac + d = 5, \; (ad + bc) = 3, \; bd = 2.$$

The last equation forces $b + d = \pm 2$ but $a = -c$ so ac is negative so the second equation can't possibly hold.

11. (15 points) Corrected Problem: Let R be a commutative ring with identity which has exactly one prime ideal P and let $D = R - P = \{r \in R \mid r \notin P\}$. Prove that R/P is a field

and that R is isomorphic to the ring of fractions $D^{-1}R$.

Since R has an identity P lies in some maximal ideal which is prime, to P is itself a maximal ideal so P is the unique maximal ideal and R/P is a field. For $r \notin P$ the ideal (r), if proper, lies in some maximal ideal so lies in P, clearly impossible. Thus $(r) = R$, so we see that D is precisely the set of units in R. Informally then inverting everything in D does nothing. More precisely define $\phi : D^{-1}R \to R$ by $\phi(\frac{r}{q} = rq^{-1}$ and check that it is an isomorphism with inverse $r \to r/1$.

12. (10 points) The quotient group G/N has order n so any element to the nth power is the identity. That is for $g \in G$ we have $(gN)^n = g^n N = N$. Thus $g^n \in N$ as desired.

13. (15 points) Let H and K be groups and $\phi : K \to \operatorname{Aut} H$ be a homomorphism. The semidirect product $H \rtimes K$ has $H \times K$ as its underlying set but with operation:

$$(h_1, k_1)(h_2, k_2) = (h_1\phi(k_1)(h_2), k_1k_2).$$

The set of (h, e) is a normal subgroup isomorphic to H with quotient group isomorphic to K.