Math 619 Final Exam - December 14, 2011

1. Short Answer- no work need be shown. (40 points)

a. Give an example of a Sylow p-subgroup in $GL_n(p)$, the $n \times n$ invertible matrices with entries in a field of p elements.

b. Find the upper and lower central series for A_4.

c. How many elements does the group $\mathrm{Aut}(S_6)$ have?

d. Give an example of a group which is solvable but not nilpotent.

e. Give an element in the symmetric group S_{15} which has maximal possible order.

f. Let $H \trianglelefteq G$. Define a *complement* for the subgroup H. Give an example where H does not have a complement.

g. Give an example of a commutative ring R and a prime ideal in R that is not maximal.

h. Give an example of a ring with exactly 6 invertible elements.

i. Give the complete subgroup lattice for the dihedral group D_8.

j. Prove or give a counterexample: $H \trianglelefteq N \trianglelefteq G$ implies $H \trianglelefteq G$.

2. (15 points) Recall that tensor products commute with direct sums. That is, for a right R-module M and left R-modules $\{N_i\}_{i\in I}$ we have:

$$M \otimes_R (\oplus_{i\in I} N_i) \cong \oplus_{i\in I}(M \otimes_R N_i).$$

Show that tensor products do not commute with direct products in general. (Hint: Consider the extension of scalars from $\mathbb{Z}$ to $\mathbb{Q}$ of the direct product of the modules $M_i = \mathbb{Z}/2^i\mathbb{Z}$.)

3. (15 points) Let I be a nilpotent ideal in a commutative ring R. Let M and N be R-modules and let $\phi : M \to N$ be an R-module homomorphism. Show that if the induced map $\overline{\phi} : M/IM \to N/IN$ is surjective, then ϕ is surjective.

4. (15 points) Let R be a commutative ring. Show that an R-module M is irreducible if and only if it is isomorphic (as an R-module) to R/I for some maximal ideal I.

5. (15 points) Let F be a field and $p(x) \in F[x]$. Describe all the ideals in the ring $F[x]/(p(x))$.

6. (10 points) Let $\phi : G \to H$ be a group homomorphism. Prove that the kernel $\ker \phi$ is a normal subgroup and $\ker \phi = \{e\}$ if and only if ϕ is injective.

7. (15 points) Let $\phi : R \to S$ be a homomorphism of commutative rings and let P be a prime ideal of S. Prove that the preimage $\phi^{-1}(P)$ is either all of R or else a prime ideal in R.

8. (10 points) Construct a field with 9 elements and give its multiplication table.

9. (15 points) Let N be a normal subgroup of G. Suppose $N \cap G' = \{e\}$. Prove that $N \subseteq Z(G)$.

10. (10 points) Prove that $p(x) = x^4 + 5x^2 + 3x + 2$ is irreducible over $\mathbb{Q}$.

11. (15 points) Let R be a commutative ring with identity which has exactly one prime ideal P and let $D = R - P = \{r \in R \mid r \notin P\}$. Prove that R/P is a field that is isomorphic to the ring of fractions $D^{-1}R$.

12. (10 points) Let G be a group and N a normal subgroup of index n. Prove that $g^n \in N$ for all $g \in G$.

13. (15 points) Describe in detail the construction of a *semidirect product* of groups.