Math 619 Fall 2008- Exam 2 Solutions

1. Let G be nonabelian of order 18. A Sylow 3 subgroup has order 9, hence index two and is normal. Since there must be a Sylow 2 subgroup or order 2, we know G is a semidirect product $P_3 \rtimes Z_2$.

Suppose first that $P_3 \cong Z_9$. The group $\operatorname{Aut} Z_9$ is abelian of order 6, hence cyclic (recall a generator of Z_9 must map to a generator). Thus $\operatorname{Aut} Z_9$ has a unique element of order 2, namely that map ψ with $\psi(x) = x^{-1}$. So we have:

$$G_1 = \langle x, y \mid x^9 = y^2 = e, yxy = x^{-1} \rangle.$$

Notice that $G_1 \cong D_{18}$.

Next suppose that $P_3 \cong Z_3 \times Z_3 = \langle a \rangle \times \langle b \rangle$. Recall that $\operatorname{Aut}(Z_3 \times Z_3) \cong GL_2(3)$ has order $(3^2 - 1)(3^2 - 3) = 48$, and so has elements of order 2. A little computation with 2×2 matrices shows that the elements $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ of order two have either $a = -d$ or $b = c = 0$. This leads to:

$$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 0 & -1 \end{pmatrix}$$

plus the transposes and negatives of the matrices above. Check that all but the first have an eigenvector of eigenvalue 1 and one of eigenvalue -1. So up to change of basis (i.e. conjugacy) there are only two conjugacy classes or matrices of order 2, namely the classes of $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ and $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$. Since Z_2 is cyclic, conjugate images give the same semidirect product. Thus we get two more groups:

$$G_2 = \langle x, a, b \mid x^2 = a^3 = b^3 = e, ab = ba, xax = a, xbx = b^{-1} \rangle \cong S_3 \times Z_3$$
$$G_3 = \langle x, a, b \mid x^2 = a^3 = b^3 = e, ab = ba, xax = a^{-1}, xbx = b^{-1} \rangle$$

2. $\{(1, 2, 3), (4, 5, 6), (7, 8, 9), (1, 4, 7)(2, 5, 8)(3, 6, 9)\}$.

3. The units in $\mathbb{Z}[i]$ are $\pm 1, \pm i$. In a DVR the units are the elements of valuation 0.

4. The center of $\mathbb{C}S_3$ is 3-dimensional. The most obvoius basis is given by the class sums, namely $\{e, (1, 2) + (1, 3) + (2, 3), (1, 2, 3) + (1, 3, 2)\}$.

5. It is a direct product of p-groups. Every maximal subgroup is normal. Every proper subgroup is proper in its normalizer. All the Sylow subgroups are normal.

6. Since $13 = (2 + 3i)(2 - 3i)$ and $(2 + 3i)$ is not in (13) then $2 + 3i + (13)$ is a zero divisor in $\mathbb{Z}[i]/(13)$. Thus it is not an integral domain, so certainly not a field.

7. Let $D = \{0 = d_0, 1 = d_1, d_2, \ldots, d_s\}$ be a finite integral domain and let $0 \neq d \in D$. If $dd_i = dd_j$ then $d_i = d_j$ since cancelation holds in an ID. Thus $\{dd_i\}$ are distinct, and since D is finite, this is

a complete set of elements of D. Thus $dd_i = 1$ for some i, so d is a unit, and hence D is a field.

8. Suppose $p \in D$ is prime. Suppose further that $p = r_1 r_2$. Since p is prime we know, WLOG, that $p \mid r_1$. Thus $r_1 = ps$ for some $s \in D$. Thus $p = psr_2$ which gives $p(1 - sr_2) = 0$. Since we are in an integral domain we have $1 - sr_2 = 0$, i.e. $sr_2 = 1$ so r_2 is a unit. Thus any factorization of p into two terms includes a unit, i.e. p is irreducible.

9. Let G have order pq with $p < q$. Then Sylow's theorem immediately implies the Sylow q subgroup Q is normal. Since it has order q it is isomorphic to Z_q and since the quotient has order p, it is isomorphic to Z_p. Thus $1 \lhd Q \lhd G$ is a composition series where each quotient is abelian, thus G is solvable by definition.

10. This problem was missing the hypothesis that n is odd. We have $D_{4n} = \langle r, s \mid r^{2n} = s^2 = e, srs = r^{-1} \rangle$. Check that $sr^n s = r^n$ so $\{e, r^n\}$ is a normal subgroup of order 2. Check also that $< s, r^2 >$ is a subgroup isomorphic to D_{2n} which is normal (since it has index 2). Since n is odd, r^n is not in this subgroup. We have two normal subgroups with trivial intersection, so their product is isomorphic to the direct product $Z_2 \times D_{2n}$. This has the same number of elements as D_{4n} so it is D_{4n}.

11. Let D be a Euclidean Domain and $I \subseteq D$ be an ideal. Choose an element $x \in I$ such that x has minimal norm among all elements of I. Let $w \in I$ so $n(w) \geq n(x)$. Since D is a ED we can write $w = qx + r$ with $n(r) < n(x)$ or $r = 0$. But $r = w - qx \in I$ so by the minimality assumption, we cannot have $n(r) < n(x)$. Thus $r = 0$ so $w = qx \in (x)$. Since $w \in I$ was arbitrary, we have shown that $I = (x)$ is principal. Thus D is a PID.