Math 619 Midterm Exam #1- October 2, 2008- SOLUTIONS

Instructions: G always denotes a *finite* group. Each of the seven problems will be weighted equally.

Part 1: Do all four problems.

1. Let P be a Sylow subgroup of G and let $N = N_G(P)$. Prove that $N_G(N) = N$.

Solution: Clearly $N \subseteq N_G(N)$. Now let $x \in N_G(N)$, so $xNx^{-1} = N$. Then both P and xPx^{-1} are Sylow p-subgroups of N, but $P \trianglelefteq N$ so is the unique Sylow p subgroup of N. Thus $xPx^{-1} = P$ so $x \in N$. Hence $N_G(N) \subseteq N$.

2. Consider $\sigma = (12)(34) \in S_4$. Write down the elements in the conjugacy class of σ. Write down the elements in the centralizer $C_{S_4}(\sigma)$.

Solution: Conjugacy class: $\{(12)(34), (13)(24), (14)(23)\}$
Centralizer: $\{e, (12), (34), (12)(34), (13)(24), (14)(23), (1324), (1423)\}$

3. Let $\phi : G \to H$ be a homomorphism and let $K \trianglelefteq H$. Prove that $\phi^{-1}\{K\}$ is a subgroup of G and that it is normal.

Solution: Note that $e \in \phi^{-1}\{K\}$ so it is nonempty. Let $x, y \in \phi^{-1}\{K\}$. Then $\phi(x), \phi(y) \in K$ so $\phi(x)\phi(y)^{-1} = \phi(xy^{-1}) \in K$ since $K \leq H$. Thus $xy^{-1} \in \phi^{-1}\{K\}$ so $\phi^{-1}\{K\}$ is a subgroup. Let $g \in G$. Then $\phi(gxg^{-1}) = \phi(g)\phi(x)\phi(g)^{-1}$ is in K since $K \trianglelefteq H$. Thus $gxg^{-1} \in \phi^{-1}\{K\}$ so $\phi^{-1}\{K\}$ is normal in G.

4. Define the following: characteristic subgroup, orbit, solvable group, commutator subgroup.

Solution: A subgroup H is *characteristic* in G if $\phi(H) = H$ for any $\phi \in \operatorname{Aut} G$. If G acts on a set A and $a \in A$ the *orbit* of a is the set $\{ga \mid g \in G\}$. A group G is *solvable* if it has a normal series where each quotient is abelian. The *commutator subgroup* of G is the subgroup generated by all commutators $\{xyx^{-1}y^{-1} \mid x, y \in G\}$.

Part 2: Choose three problems. Clearly indicate which three you have chosen.

5. Let G have odd order and let $g \in G$ be a nonidentity element. Prove that g is not conjugate to g^{-1}.

Solution: G has odd order so it has no elements of order two, thus no nonidentity element is equal to its inverse. Suppose $g = xg^{-1}x^{-1}$. Suppose also that h is conjugate to g, so $h = aga^{-1}$. Then $h^{-1} = ag^{-1}a$ so h^{-1} is conjugate to g^{-1} and hence to g and h. Thus for each h in the conjugacy class of g, $h^{-1} \neq h$ is also in the class, so the class has even order. This is a contradiction since the size of the conjugacy class is the index of the centralizer of g; in particular it must divide the order of the group.

6. a. Show that $\operatorname{Inn} S_4 \cong S_4$.

b. Write down the Sylow 3 subgroup(s) of S_4.

c. Suppose $\phi \in \operatorname{Aut} S_4$. Show that ϕ acts on the Sylow 3 subgroups and that any ϕ which fixes all of them must be the identity.

d. Use c to define an injective homomorphism from $\operatorname{Aut} S_4$ to S_4. Conclude that $\operatorname{Aut} S_4 \cong S_4$.

Solution: a. We showed in class that $Z(S_4) = \{e\}$ so $\operatorname{Inn} S_4 \cong S_4/Z(S_4) \cong S_4$.

b. $P_1 = \{e, (123), (132)\}$, $P_2 = \{e, (124), (142)\}, P_3 = \{e, (134), (143)\}, P_4 = \{e, (234), (243)\}$.

c. Any automorphism preserves the order of subgroups, so permutes the Sylow 3-subgroups. We know $\phi(x) = x^{-1}$ is not an automorphism, since S_4 is nonabelian. So choose ϕ preserving the Sylows and assume WLOG that $\phi(123) = (123)$ and $\phi(124) = (142)$. Then $(123)(124) = (13)(24)$ and $(123)(142) = (143)$. Thus $\phi(13)(24) = (143)$ which is a contradiction, since $(13)(24)$ has order 2 and (143) has order 3. Any other pair of 3-cycles works the same way, so it must be that ϕ fixes all the three-cycles. These generate S_4 so ϕ is the identity.

d. $\operatorname{Aut} S_4$ acts on the four Sylows so we get a homomorphism from $\operatorname{Aut} S_4$ to S_4. The kernel is trivial by *c*, so $\operatorname{Aut} S_4$ is isomorphic to a subgroup of S_4. Part *a* now tells us $\operatorname{Aut} S_4 \cong \operatorname{Inn} S_4 \cong S_4$.

7. Let P be a normal Sylow p-subgroup of G and let $H \leq G$. Prove that $P \cap H$ is the unique Sylow p-subgroup of H.

Solution: $P \cap H \trianglelefteq H$ and the second isomorphism theorem says that $H/P \cap H \cong PH/P$. Since P is a Sylow of G it is clear that p does not divide $[PH : P] = [H : P \cap H]$. But $P \cap H$ is a p-subgroup, so it is a Sylow subgroup of H. Since it is normal in H, it is the unique Sylow p-subgroup.

8. Let $\mathbf{F}_p$ denote the field with p elements and let $G = GL_2(\mathbf{F}_p)$.
a. What is the order of G?

b. Prove that G has $p + 1$ Sylow p-subgroups. (Hint: The Sylow theorems will show you must only produce two distinct Sylow subgroups)

Solution:
a. G has order $(p^2 - 1)(p^2 - p) = p(p + 1)(p - 1)^2$

b. The order of G tells us the Sylows have size p. The number n_p is congruent to 1 mod p and divides $(p + 1)(p - 1)^2$. One easily sees the only possibilities are 1 or $p + 1$. But $\{\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}\}$ and $\{\begin{pmatrix} 1 & 0 \\ a & 1 \end{pmatrix}\}$ are both subgroups of order p, so there are at least two Sylows, and thus there are $1 + p$ Sylows.

9. Let G be a non-abelian p-group of order p^3, where p is a prime number. Let $Z(G)$ be the center of G and G' be its commutator subgroup.

a. Show that $Z(G) = G'$ and that this is the unique normal subgroup of G of order p.

b. Determine the number of distinct conjugacy classes of G.

Solution: a. G is nonabelian so $G/Z(G)$ cannot be cyclic, so cannot have order p. Thus $Z(G)$ must have order p. Now $G/Z(G)$ has order p^2 and is thus abelian. Hence $G' \leq Z(G)$. However $G' \neq \{e\}$ since G is nonabelian, so $G' = Z(G)$. Now for any normal subgroup H of order p we must have G/H of order p^2 hence abelian hence $G' \leq H$. Thus G' is the unique normal subgroup of order p.

b. For $g \in Z(G)$ the class of g is just g, so there are p classes of size one. For $g \notin Z(G)$ the centralizer $C_G(g)$ includes all of $< g >$ and $Z(G)$ so has order p^2. Thus the conjugacy class has order $p^3/p^2 = p$. Hence there are p classes of size one and $(p^3 - p)/p = p^2 - 1$ classes of size p for a total of $p^2 + p - 1$ conjugacy classes.