Math 619-Final Exam Fall 2008- SOLUTION SKETCH

You may use your class notes and homework assignments, as well as Dummit and Foote, but no other material of any sort. Do not discuss the exam with anyone but the instructor. **The exam is due on Monday December 15 at noon.**

Do all the problems.

1. Let R be commutative and I, J ideals of R. Prove that

$$R/I \otimes_R R/J \cong R/(I+J)$$

as R modules.

Solution: Define a map $\psi : R/I \times R/J \to R/(I+J)$ by $\psi(r_1 + I, r_2 + J) = r_1 r_2 + I + J$. Since $I, J \subset I + J$, ψ is well-defined. Check that ψ is R-bilinear, so induces a map $\tilde{\psi} : R/I \otimes R/J \to R/(I+J)$ given by $\tilde{\psi}(r_1 + I \otimes r_2 + J) = r_1 r_2 + I + J$, which is also an R-module homomorphism. Define a map Φ in the other direction by $\Phi(r+I+J) = r \otimes 1 = 1 \otimes r$. Notice that $\Phi(I) = \Phi(J) = 0$, so $\Phi(I+J) = 0$ so Φ is well-defined. Check that Φ is also a module homomorphism and $\Phi \circ \tilde{\psi}$ and $\tilde{\psi} \circ \Phi$ are the respective identity maps, and so Φ and $\tilde{\psi}$ thus are isomorphisms.

2. Let R and S be rings and let ${}_RM$ be a left R module and ${}_RN_S$ an R-S bimodule.

a. Prove that $\mathrm{Hom}_R(M, N)$ is a right S module with the action of S given by $(fs)(m) = f(m)s$.

Solution: First we check that $fs \in \mathrm{Hom}_R(M, N)$. We have $(fs)(rm) = f(rm)s = rf(m)s = r(fs)(m)$ as desired. Now to check that we have a right S module we must verify that $(f_1 + f_2)s = f_1 s + f_2 s$ and $f(s_1 + s_2) = fs_1 + fs_2$. Both of these are immediate. Finally notice that $f(s_1 s_2)(m) = f(m)s_1 s_2$ and $(fs_1)s_2(m) = fs_1(m)s_2 = f(m)s_1 s_2$ so $f(s_1 s_2) = (fs_1)s_2$.

b. Suppose $\psi :_R M \to_R M'$ is a homomorphism of left R modules. Notice that ψ induces a map:

$$\tilde{\psi} : \mathrm{Hom}_R(M', N) \to \mathrm{Hom}_R(M, N)$$

given by $\tilde{\psi}(f) = f \circ \psi$. Prove that $\tilde{\psi}$ is a homomorphism of right S modules.

Solution: Easy to check that $\tilde{\psi}(f_1 + f_2) = \tilde{\psi}(f_1) + \tilde{\psi}(f_2)$ so we have a homomorphism of abelian groups. Now let $s \in S$ and $m \in M$. Then

$$\begin{aligned} \tilde{\psi}(fs)(m) &= (fs) \circ \psi(m) \\ &= f(\psi(m))s \\ &= \tilde{\psi}(f)(m)(s) \\ &= (\tilde{\psi}(f))s(m). \end{aligned}$$

Thus $\tilde{\psi}(fs) = (\tilde{\psi}(f))s$ so $\tilde{\psi}$ is a right S-module map.

3. Suppose $f : M \to N$ and $g : N \to M$ are R-module homomorphisms such that $g \circ f = 1_M$. Prove that $N \cong \mathrm{Im}\, f \oplus \mathrm{Ker}\, g$.

Solution: Let $n \in N$. Then $n = f(g(n)) + n - f(g(n))$. Notice that $g(n - f(g(n))) = g(n) - g \circ f \circ g(n) = 0$ since $g \circ f = 1$. Thus $n \in \mathrm{Im}\, f + \mathrm{Ker}\, g$ so $N = \mathrm{Im}\, f + \mathrm{Ker}\, g$. Now suppose $f(m) \in \mathrm{Im}\, f \cap \mathrm{Ker}\, g$. Then $g(f(m)) = 0$ but $g(f(m)) = m$ so $m = 0$ and thus $f(m) = 0$. We have shown N is the sum and since the intersection is 0, the sum is direct.

4. Show that the ring R of even integers contains a maximal ideal I such that R/I is not a field.

Solution: Let $I = (4) = 4\mathbb{Z}$ which is clearly an ideal in R. Then R/I has two elements, namely $0+I$ and $2+I$. But both elements square to 0 in the ring R/I so R/I is not a field. Suppose $a \notin I$. Then a is even but not a multiple of 4 so the gcd of a and 4 is two. Thus the subgroup generated by a and 4 is $2\mathbb{Z} = R$. This proves I is a maximal ideal (actually more, that it is a maximal subgroup).

5. Page 213 #7: Prove there are no simple groups of order 1755 or 5265. (Some reading of the beginning of Section 6.2 will be helpful here.)

Solution: Suppose G is simple of order $1755 = 3^3 * 5 * 13$. Then $n_3 \equiv 1 \bmod 3$ and $n_3 \mid 65$ which forces $n_3 = 13$ since $n_3 \neq 1$ by assumption. Then G acts transitively on these 13 Sylow 3-subgroups by conjugation, which gives a nontrivial homomorphism $\psi : G \to S_{13}$. The kernel is normal, so trivial, and thus ψ is an injection, so G is isomorphic to a subgroup of S_{13}, so WLOG assume $G \leq S_{13}$. Let P_{13} be a Sylow subgroup of G (and hence of S_{13}) so P is generated by a 13-cycle. The Sylow theorems tell us that $n_{13} = 27$ so $[G : N_G(P)] = 27$ so $|N_G(P)| = 65$. We have:

$$Z_{13} \cong P \leq G \leq S_{13}.$$

Then $N_G(P) \leq N_{S_{13}}(P)$, so $|N_G(P)|$ divides $|N_{S_{13}}(P)| = 13 * 12$. But 65 does not divide 13*12, so we have a contradiction.

The proof for $5265 = 3^4 * 5 * 13$ is identical, except we conclude $|N_G(P)| = 195$.

6. How many Sylow p subgroups does the general linear group $GL_3(p)$ have?

Solution: Let P be a Sylow p subgroup. Then $n_p = [G : N_G(P)]$. Recall that

$$|GL_3(p)| = (p^3 - 1)(p^3 - p)(p^3 - p^2) = p^3(p-1)^3(p^2 + p + 1)(p + 1)$$

and that the upper unitriangular matrices are a Sylow p-subgroup of order p^3.

Let $p = \begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix} \in P$ be arbitrary. Suppose $T \in N_G(P)$. This is true if and only if $TpT^{-1} \in P$ for any $p \in P$, i.e. if and only if $Tp = \tilde{p}T$ for some $\tilde{p} \in P$. So suppose:

$$T = \begin{pmatrix} d & e & f \\ g & h & i \\ j & k & l \end{pmatrix} \in N_G(P), \;\; \tilde{p} = \begin{pmatrix} 1 & A & B \\ 0 & 1 & C \\ 0 & 0 & 1 \end{pmatrix}$$

Set $Tp = \tilde{p}T$ and assume a, b, c are nonzero. The matrix equality gives us 9 equations in the entries of $T, p, \tilde{p}$, from which one easily solves that $g = j = k = 0$ so T is upper triangular. But it is easy to see upper triangular matrices are all in the normalizer. Thus:

$$N_G(P) = \left\{ \begin{pmatrix} d & e & f \\ 0 & h & i \\ 0 & 0 & l \end{pmatrix} \mid dhl \neq 0 \right\}.$$

We conclude that $|N_G(P)| = (p-1)^3 p^3$ so $n_p = (p^2 + p + 1)(p + 1)$.

7. Let G be infinite and let $H < G$ be a subgroup of finite index. Prove that G is not simple. Hint: p.122 #8.

Solution: Suppose $[G : H] = n$. Then G acts on the left cosets G/H which gives a nontrivial homomorphism $\psi : G \to S_n$. The kernel is a subgroup with $G/K \cong \psi(G) \leq S_n$. Thus G/K is finite so K is normal of finite index so $K \neq 1$ and G is not simple.

8. Recall that the commutator of x and y is $[x, y] = xyx^{-1}y^{-1}$. Notice in an abelian group that the identity is the only commutator. Prove, more generally, that if $[G : Z(G)] = n$ that G has at most n^2 different commutators.

Solution: Suppose $x_1Z(G) = x_2Z(G)$ and $y_1Z(G) = y_2Z(G)$ so $x_1 = x_2z$ and $y_1 = y_2\tilde{z}$ for some $z, \tilde{z} \in Z(G)$. Check that $[x_1, y_1] = [x_2, y_2]$. Thus $[x, y]$ is determined completely by the cosets $xZ(G)$ and $yZ(G)$ so there are at most n^2 possibilities.

9. Let $\mathbb{Z}[\frac{1}{2}]$ be the subring of $\mathbb{Q}$ generated by $\mathbb{Z}$ and $\frac{1}{2}$. Prove or disprove that $\mathbb{Z}[\frac{1}{2}]$ is a free $\mathbb{Z}$ module.

Solution: Check that $\mathbb{Z}[\frac{1}{2}]$ is just the set of rational numbers which, in lowest terms, have denominator a power of 2. Suppose it were a free abelian group. Suppose further the rank is at least 2, so there is a basis $\{b_1, b_2, \dots\}$ where $b_1 = \frac{m}{2^i}$ and $b_2 = \frac{n}{2^j}$ with m and n odd and $i, j \geq 0$. Assume that $i \geq j$ WLOG. Then:

$$\frac{mn}{2^j} = mb_2 = 2^{i-j}nb_1$$

and both m and $2^{i-j}n$ are in $\mathbb{Z}$. This contracts the freeness of the basis. Thus it must be rank 1, i.e. $\cong \mathbb{Z}$. But this is clearly impossible. The subgroup generated by $\frac{a}{2^i}$ does not have elements with denominators having powers of 2 greater than 2^i, so $\mathbb{Z}[\frac{1}{2}]$ is not cyclic, so not isomorphic to $\mathbb{Z}$.

10. Let $\mathbb{Z}_7$ denote the ring of integers modulo 7. Let $p(x) = x^3 + 2x + 1 \in \mathbb{Z}_7[x]$.

a. Either prove $p(x)$ is irreducible or exhibit a factorization of it.

b. How many elements are in the ring $\mathbb{Z}_7[x]/(p(x))$? Is it a field? Explain.

Solution: $p(x)$ is degree 3 so it is irreducible if and only if it has no roots. Check that $p(0), p(1), \dots, p(6)$ are all nonzero, so $p(x)$ is irreducible. Thus $\mathbb{Z}_7[x]/(p(x))$ is a field. Every coset is uniquely written as $a_0 + a_1x + a_2x^2 + (p(x))$ with $a_0, a_1, a_2 \in Z_7$. Thus it is a field of 7^3 elements.