Math 619-Final Exam Fall 2008

You may use your class notes and homework assignments, as well as Dummit and Foote, but no other material of any sort. Do not discuss the exam with anyone but the instructor. Assume rings have identity except in Problem 4.**The exam is due on Monday December 15 at noon.**

Do all the problems.

1. Let R be commutative and I, J ideals of R. Prove that

$$R/I \otimes_R R/J \cong R/(I+J)$$

as R modules.

2. Let R and S be rings and let ${}_RM$ be a left R module and ${}_RN_S$ an R-S bimodule.

a. Prove that $\mathrm{Hom}_R(M,N)$ is a right S module with the action of S given by $(fs)(m) = f(m)s$.

b. Suppose $\psi :_R M \to_R M'$ is a homomorphism of left R modules. Notice that ψ induces a map:

$$\tilde{\psi} : \mathrm{Hom}_R(M', N) \to \mathrm{Hom}_R(M, N)$$

given by $\tilde{\psi}(f) = f \circ \psi$. Prove that $\tilde{\psi}$ is a homomorphism of right S modules.

3. Suppose $f : M \to N$ and $g : N \to M$ are R-module homomorphisms such that $g \circ f = 1_M$. Prove that $N \cong \operatorname{Im} f \oplus \operatorname{Ker} g$.

4. Show that the ring R of even integers contains a maximal ideal I such that R/I is not a field.

5. Page 213 #7: Prove there are no simple groups of order 1755 or 5265. (Some reading of the beginning of Section 6.2 will be helpful here.)

6. How many Sylow p subgroups does the general linear group $GL_3(p)$ have?

7. Let G be infinite and let $H < G$ be a subgroup of finite index. Prove that G is not simple. Hint: p.122 #8.

8. Recall that the commutator of x and y is $[x,y] = xyx^{-1}y^{-1}$. Notice in an abelian group that the identity is the only commutator. Prove, more generally, that if $[G : Z(G)] = n$ that G has at most n^2 different commutators.

9. Let $\mathbb{Z}[\frac{1}{2}]$ be the subring of $\mathbb{Q}$ generated by $\mathbb{Z}$ and $\frac{1}{2}$. Prove or disprove that $\mathbb{Z}[\frac{1}{2}]$ is a free $\mathbb{Z}$ module.

10. Let $\mathbb{Z}_7$ denote the ring of integers modulo 7. Let $p(x) = x^3 + 2x + 1 \in \mathbb{Z}_7[x]$.

a. Either prove $p(x)$ is irreducible or exhibit a factorization of it.

b. How many elements are in the ring $\mathbb{Z}_7[x]/(p(x))$? Is it a field? Explain.