Math 561 Fall 2013 Homework Number 1

Due Monday September 9, 2013

1. In this exercise we classify the complex irreducible representations for the group S_3, which is generated by $\tau = (1,2,3)$ and $\sigma = (1,2)$, following 1.10 in Fulton-Harris. Th ad-hoc technique is to restrict to the cyclic subgroup $\langle \tau \rangle$.

a) Let $\mathbb{C}^3$ with natural basis $\{e_1, e_2, e_3\}$ be an S_3 module via $\rho \cdot e_i = e_{\rho(i)}$. The line spanned by $(1,1,1)$ is invariant, and has a complementary subspace:

$$V := \{(z_1, z_2, z_3) \mid z_1 + z_2 + z_3 = 0.\}$$

Verify that V is an irreducible two-dimensional submodule of $\mathbb{C}^3$; it is called the *standard representation* of S_3.

b) Let $\omega = e^{\frac{2\pi i}{3}}$ be a primitive cube root of unity. Find a basis for $\{\alpha, \beta\}$ for V so that $\tau\alpha = \omega\alpha$, $\tau\beta = \omega^2\beta$, $\sigma\alpha = \beta, \sigma\beta = \alpha$.

c) Now let W be an arbitrary finite-dimensional $\mathbb{C}S_3$ module and let v be an eigenvector for τ with eigenvalue ω^i. Use the relation $\tau\sigma = \sigma\tau^2$ to conlude that $\sigma(v)$ is an eigenvector for τ with eigenvalue ω^{2i}.

d) In the previous part, if $i = 1, 2$, prove that $\{v, \sigma(v)\}$ span a submodule of W isomorphic to V. If $i = 0$ then consider two cases, depending whether $\sigma(v)$ is a multiple of v or not. In the first case either $\sigma(v) = v$ or $\sigma(v) = -v$, so we see two possible one-dimensional submodules. If not, $\{v, \sigma(v)\}$ is two dimensional. Show it is a direct sum of one-dimensional modules spanned by $v + \sigma(v)$ and $v - \sigma(v)$.

e) Use that W is a direct sum of its eigenspaces (proof later) to conlude that $\mathbb{C}S_3$ has precisely three isomorphism classes of irreducible modules, two one-dimensional modules together with V

2. Define permutations $a, b, c \in S_6$ by:

$$a = (1,2,3), b = (4,5,6), c = (2,3)(4,5)$$

and let G be the subgroup of S_6 generated by a, b, c.

a) Check the relations $a^3 = b^3 = c^2 = 1, ab = ba, ac = ca^{-1}, bc = cb^{-1}$.

b) Use these relations to deduce that G has 18 elements.

c) Let η and ν be complex cube roots of unity (e.g. 1, ω or ω^2 from Prob 1). Prove (by checking relations) there is a representation ρ of G so that:

$$\rho(a) = \begin{pmatrix} \epsilon & 0 \\ 0 & \epsilon^{-1} \end{pmatrix}, \rho(b) = \begin{pmatrix} \nu & 0 \\ 0 & \nu^{-1} \end{pmatrix}, \rho(c) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

d) For which values of ϵ, ν is ρ faithful? Irreducible?