Math 561 Midterm Exam Solutions

October 18, 2013

1a. $[x,[y,z]]+[y,[z,x]]+[z,[x,y]]=0$ for all $x,y,z,\in\mathfrak{g}$.

b. Each vertex v is assigned a vector space $W(v)$ and for each edge e pointing from v_1 to v_2 there is a linear map $T_e : W(v_1)\to W(v_2)$.

c. The group acts diagonally $g(u\otimes v)=gu\otimes gv$ and the action is extended to kG by linearity.

d. The radical is the common annihilator of all simple A modules, i.e. $\mathrm{rad}(A)=\{a\in A \mid aS = 0 \text{ for all simple A- modules } S\}$.

e. Let $\chi_1,\dots,\chi_r$ be the irreducible characters. Then

$$\sum_{i=1}^{r}\chi_i(g)\overline{\chi_i(h)}=\begin{cases}|C_G(g)| & \text{if } g \text{ and } h \text{ are conjugate,}\\ 0 & \text{else.}\end{cases}$$

f. Let $\{g_i\}_{i=1}^{t}$ be a set of left coset representatives. Then:

$$(\mathrm{Ind}_H^G\,\psi)(g)=\sum_{i=1}^{t}\dot{\psi}(g_i^{-1}gg_i)$$

where $\dot{\psi}(x)$ is defined to be zero if $x\notin H$.

g. $FS(\chi)=\frac{1}{|G|}\sum_{g\in G}\chi(g^2)$.

h. The universal enveloping algebra is the tensor algebra $T(\mathfrak{g})$ quotiented out by the ideal generated by $\{xy-yx-[x,y]\mid x,y\in\mathfrak{g}\}$.

2a. The order is 108, the sum of the character degrees.

b. Using column orthogonality to calculate centralizer orders, we get the conjugacy class sizes are 1, 9, 9, 6, 6, 2, 9, 18, 18, 12, 18.

c. The center is trivial, there are no other classes of size 1.

d. The number of linear characters is $4 = |G/G'|$ so G' has 27 elements.

e. Calculating the kernels of the irreducibles we see that χ_{10} and χ_{11} have trivial kernel. Every other irreducible has C_1 and C_6 in the kernel so $C_1 \cup C_6$ is the smallest normal subgroup, it has order 3.

f. The kernel of $X.6$ is $C_1 \cup C_3 \cup C_4 \cup C_6$. The irreducible characters of G/K correspond to those of G which have K in their kernel. These are $X.1, X.3, X.6$. Focusing on these 3 we get 3 distinct columns and so the character table if G/K is:

$$\begin{matrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 2 & 0 & -1 \end{matrix}$$

This is a nonabelian group of order 6, so $G/K \cong S_3$.

g. $X.5 + X.6 + X.9$

3. Let $H \leq G$ and let ψ be a character of H and χ a character of G. Then:

$$(\psi, \chi_H) = (\psi^G, \chi).$$

Thus to get the decomposition we need to restrict each of $\psi_1, \ldots, \psi_7$ to A_4 and calculate the inner product with X_4. Doing so we obtain $\psi_3 + \psi_4 + \psi_5 + \psi_6 + 2\psi_7$.