# Math 561 Midterm Exam

October 18, 2013

**1. Short Answer- no work need be shown. (48 points)**

**a)** Give the Jacobi identity for Lie algebras.

**b)** Let $Q$ be a quiver with vertices $V$ and edges $E$. Define a representation of $Q$.

**c)** Let $G$ be a group and $U, V$ be $kG$ modules. How does $kG$ act on the tensor product $U \otimes V$?

**d)** Define the radical of an algebra $A$.

**e)** State the column orthogonality relation for a character table.

**f)** Let $H \leq G$ and $\psi$ a character of $H$. Give a formula for the induced character $(\mathrm{Ind}_H^G \psi)(g)$.

**g)** Define the Frobenius-Schur indicator of a character.

**h)** Let $\mathfrak{g}$ be a Lie algebra. Define the *universal enveloping algebra* $\mathcal{U}(\mathfrak{g})$.

**2. (35 points)** Consider the character table below for an unknown group $G$. Label the conjugacy classes $C1, C2, C3, \ldots, C11$ corresponding to the 11 columns.

| | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| X.1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| X.2 | 1 | -1 | -1 | 1 | 1 | 1 | 1 | -1 | -1 | 1 | 1 |
| X.3 | 1 | -1 | 1 | 1 | 1 | 1 | -1 | -1 | 1 | 1 | -1 |
| X.4 | 1 | 1 | -1 | 1 | 1 | 1 | -1 | 1 | -1 | 1 | -1 |
| X.5 | 2 | . | -2 | 2 | -1 | 2 | . | . | 1 | -1 | . |
| X.6 | 2 | . | 2 | 2 | -1 | 2 | . | . | -1 | -1 | . |
| X.7 | 2 | -2 | . | -1 | 2 | 2 | . | 1 | . | -1 | . |
| X.8 | 2 | 2 | . | -1 | 2 | 2 | . | -1 | . | -1 | . |
| X.9 | 4 | . | . | -2 | -2 | 4 | . | . | . | 1 | . |
| X.10 | 6 | . | . | . | . | -3 | -2 | . | . | . | 1 |
| X.11 | 6 | . | . | . | . | -3 | 2 | . | . | . | -1 |

**a)** Determine the order of $G$.

**b)** Determine the size of each conjugacy class.

**c)** Determine the center of $G$ as a union of conjugacy classes.

**d)** Determine the order of the commutator subgroup $G'$.

**e)** What is the order of the smallest nontrivial normal subgroup of $G$?

**f)** Let $K = \ker(X.6)$. Calculate the character table of $G/K$ and determine its isomorphism class.

**g)** Decompose the tensor product $X.8 \cdot X.9$ into irreducibles.

**3. (17 points)** Below are the character tables for $A_4$ and $S_5$ respectively, with conjugacy class sizes shown, and where $w$ is a primitive cube root of unity. State the Frobenius reciprocity result for characters, and then use Frobenius reciprocity to decompose $\mathrm{Ind}_{A_4}^{S_5} \chi_4$ into irreducible $S_5$ characters.

| | 1 | 10 | 15 | 20 | 30 | 24 | 20 |
|---|---|---|---|---|---|---|---|
| | E | (1,2) | (1,2)(3,4) | (1,2,3) | (1,2,3,4) | (1,2,3,4,5) | (1,2)(3,4,5) |
| $\psi_1$ | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| $\psi_2$ | 1 | -1 | 1 | 1 | -1 | 1 | -1 |
| $\psi_3$ | 4 | 2 | 0 | 1 | 0 | -1 | -1 |
| $\psi_4$ | 4 | -2 | 0 | 1 | 0 | -1 | 1 |
| $\psi_5$ | 5 | 1 | 1 | -1 | -1 | 0 | 1 |
| $\psi_6$ | 5 | -1 | 1 | -1 | 1 | 0 | -1 |
| $\psi_7$ | 6 | 0 | -2 | 0 | 0 | 1 | 0 |

| | 1 | 4 | 4 | 3 |
|---|---|---|---|---|
| | E | (1,2,3) | (1,3,2) | (1,2)(3,4) |
| $X_1$ | 1 | 1 | 1 | 1 |
| $X_2$ | 1 | w | w^2 | 1 |
| $X_3$ | 1 | w^2 | w | 1 |
| $X_4$ | 3 | 0 | 0 | -1 |