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A Probabilistic Proof of a Formula for the Number of Young Tableaux of a Given Shape

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1. INTRODUCTION

This paper contains a short proof of a formula by Frame, Robinson, and Thrall [1] which counts the number of Young tableaux of a given shape.

Let $\lambda = \{\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_m\}$ be a partition of n. The *Ferrers diagram* of λ is an array of cells doubly indexed by pairs (i, j) with $1 \leqslant i \leqslant m$, $1 \leqslant j \leqslant \lambda_i$. A *Young tableau of shape* λ (sometimes called a *standard tableau*) is an arrangement of the integers $1, 2, \ldots, n$ in the cells of the Ferrers diagram of λ such that all rows and columns form increasing sequences. The total number of Young tableaux of shape λ will be denoted by f_λ.

For each cell (i, j) define the *hook* H_{ij} to be the collection of cells (a, b) such that $a = i$ and $b \geqslant j$ or $a \geqslant i$ and $b = j$. Define the *hook length* h_{ij} to be the number of cells in H_{ij}.

THEOREM 1 (Frame–Robinson–Thrall [1]). *If λ is a partition of n, then*

$$f_\lambda = \frac{n!}{\prod h_{ij}},$$

where the product is over all cells in the Ferrers diagram of λ.

For example, if $\lambda = \{3, 2\}$ and $n = 5$, the hook lengths of each cell in the Ferrers diagram of λ are as shown:

$$\begin{matrix} 4 & 3 & 1 \\ 2 & 1 & \end{matrix}$$

According to Theorem 1, the number of Young tableaux of shape λ is $5!/4 \cdot 3 \cdot 1 \cdot 2 \cdot 1 = 5$, a result which can be checked easily.

Surprisingly, in view of the simplicity of the formula, it is difficult to explain why the hook lengths h_{ij} appear. They do not seem to be involved naturally in

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any direct combinatorial correspondence. In the original proof [1] hook lengths appear in the course of rearranging terms in another formula for f_λ due independently to Young [7] and to Frobenius [2]. The original proofs of the latter formula use complicated algebraic methods (group characters, symmetric polynomials). Another proof of the Young–Frobenius formula was found by MacMahon [5], using difference methods. The reader is referred to [4] for a good exposition of these arguments. A more recent proof, due to Hillman and Grassl [3] gives perhaps the best combinatorial "explanation" of hooks to date. Their proof uses hooks in a natural way to derive a combinatorial correspondence involving plane partitions, from which the Frame–Robinson–Thrall formula can be derived by an asymptotic argument.

In this paper, we give a short direct proof. The key step is based on a probabilistic model in which hooks appear in an essential way. The method also yields an algorithm which chooses, uniformly at random, a Young tableau of given shape.

2. Proof of the Formula

The first steps are the same as those found in [5]. (See [4].) Define a function

$$\begin{aligned} F(\lambda_1, \lambda_2, \ldots, \lambda_m) &= \frac{n!}{\prod h_{ij}} \qquad \text{if} \quad \lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_m, \\ &= 0, \qquad\qquad \text{otherwise}. \end{aligned}$$

In any standard tableau, the integer n must appear at a "corner," i.e., a cell which is at the end of some row and, simultaneously, at the end of a column. Removing this cell leaves a Young tableau of smaller shape. Thus the Frame–Robinson–Thrall formula follows by induction if it can be shown that

$$F(\lambda_1, \lambda_2, \ldots, \lambda_m) = \sum_\alpha F(\lambda_1, \lambda_2, \ldots, \lambda_{\alpha-1}, \lambda_\alpha - 1, \lambda_{\alpha+1}, \ldots, \lambda_m),$$

which we abbreviate

$$F = \sum_\alpha F_\alpha .$$

(Note that the summation is, in effect, over all corners, since terms for which $\lambda_{\alpha+1} > \lambda_\alpha - 1$ are zero.) Our proof consists of verifying the identity

$$1 = \sum_\alpha \frac{F_\alpha}{F}$$

by giving it a probabilistic interpretation.

Consider the following experiment: A cell (i, j) in the Ferrers diagram of λ is chosen at random, with uniform probability $1/n$. Another cell $(i', j') \neq (i, j)$ is chosen at random from among the remaining cells in the hook H_{ij}, with uniform probability $1/(h_{ij}-1)$. A new cell is chosen at random from the remaining cells in the hook $H_{i'j'}$, and so on. The process continues until a corner cell (α, β) is reached, where the process stops. This completes a single trial. The cell (α, β) where the process stops will be called the *terminal cell* of the trial.

Any corner cell (α, β) can be the terminal cell of a trial. Let $p(\alpha, \beta)$ denote the probability that a random trial terminates in cell (α, β).

THEOREM 2. *Let* (α, β) *be a corner cell. Then*

$$p(\alpha, \beta) = F_\alpha / F.$$

Proof. An easy calculation shows that

$$\begin{aligned} \frac{F_\alpha}{F} &= \frac{1}{n} \prod_{1 \leqslant i < \alpha} \frac{h_{i\beta}}{h_{i\beta}-1} \prod_{1 \leqslant j < \beta} \frac{h_{\alpha j}}{h_{\alpha j}-1} \\ &= \frac{1}{n} \prod_{1 \leqslant i < \alpha} \left(1 + \frac{1}{h_{i\beta}-1}\right) \prod_{1 \leqslant j < \beta} \left(1 + \frac{1}{h_{\alpha j}-1}\right) \end{aligned} \tag{1}$$

(with the convention that empty products are equal to 1). Our object will be to interpret each term in the expansion of these products.

Let P: $(ab) = (a_1 b_1) \to (a_2 b_2) \to \cdots \to (a_m b_m) = (\alpha\beta)$ be the path determined by a trial which begins at (ab) and ends at $(\alpha\beta)$. Define the *vertical* and *horizontal projections of* P to be the sets $A = \{a_1, a_2, ..., a_m\}$ and $B = \{b_1, b_2, ..., b_m\}$. Let $p(A, B \mid a, b)$ denote the probability that a random trial which begins at (ab) has vertical and horizontal projections A and B. We need the following:

LEMMA 3.

$$p(A, B \mid a, b) = \prod_{\substack{i \in A \\ i \neq \alpha}} \frac{1}{h_{i\beta}-1} \cdot \prod_{\substack{j \in B \\ j \neq \beta}} \frac{1}{h_{\alpha j}-1}$$

Proof of Lemma. Trivially,

$$p(A, B \mid a, b) = \frac{1}{h_{ab}-1} \{p(A - a_1, B \mid a_2, b_1) + p(A, B - b_1 \mid a_1, b_2)\}.$$

By induction on m we may assume that

$$p(A - a_1, B \mid a_2, b_1) = (h_{a_1\beta} - 1) \cdot \prod$$

and

$$p(A, B - b_1 \mid a_1, b_2) = (h_{\alpha b_1} - 1) \cdot \prod,$$

where $\prod$ is the right-hand side of the statement of Lemma 3. Thus

$$p(A, B \mid a, b) = \frac{1}{h_{ab} - 1} \{(h_{a_1\beta} - 1) + (h_{\alpha b_1} - 1)\} \cdot \prod.$$

But $h_{ab} - 1 = (h_{a_1\beta} - 1) + (h_{\alpha b_1} - 1)$, as an easy argument shows. Thus $p(A, B \mid a, b) = \prod$, as desired.

Continuing with the proof of Theorem 3, the probability $p(\alpha\beta)$ can be computed by summing the conditional probabilities with respect to the first cell chosen, and then, for each such first cell, summing over all possible vertical and horizontal projections. Thus

$$p(\alpha\beta) = \frac{1}{n} \sum p(A, B \mid a, b),$$

where the sum is over all A, B, a, b for which $A \subseteq \{1, 2, \ldots, \alpha\}$ and $B \subseteq \{1, 2, \ldots, \beta\}$, and $a = \min A$ and $b = \min B$. By Lemma 3, this is the same as expanding the products which appear on the right side of (1), and the proof is complete.

Corollary 4. $\sum_\alpha (F_\alpha / F) = 1$.

Proof. Every trial stops at some terminal cell. Thus the probabilities $p(\alpha\beta)$ must add up to 1.

As noted before, this completes the proof of Theorem 1.

3. Further Remarks

(i) The random process described here gives a very efficient method for constructing random Young tableaux of a given shape λ. If λ is a partition of n, consider a sequence of n trials, in which the terminal cell has been removed after each trial (and the probabilities $1/(h_{ij} - 1)$ revised according to the new shape). Label the first cell removed with the integer n, the next with the integer $n - 1$, and so on. Trivially, the result is a Young tableau of shape λ, and it is not hard to see (by induction, using Theorem 2) that all tableaux of shape λ appear with equal probability.

The reader is referred to [6] for further discussion of this and related results.

(ii) It is interesting to calculate the probability $p(\alpha\beta \mid ab)$ that cell $(\alpha\beta)$ will be the terminal cell, given that (ab) is the initial cell. Evidently

$$p(\alpha\beta \mid ab) = \sum_{A,B} p(A, B \mid a, b)$$

summed over all $A \subseteq \{a, a+1,..., \alpha\}$ and $B \subseteq \{b, b+1,..., \beta\}$ such that $a = \min A$ and $b = \min B$. By Lemma 3, this can be expressed as

$$\left\{\frac{1}{h_{a\beta}-1} \prod_{a<i<\alpha} \left(1+\frac{1}{h_{i\beta}-1}\right)\right\} \cdot \left\{\frac{1}{h_{\alpha b}-1} \prod_{b<j<\beta} \left(1+\frac{1}{h_{\alpha j}-1}\right)\right\},$$

provided that $a < \alpha$ and $b < \beta$. (If $a = \alpha$ the first factor is omitted, and similarly the second is omitted if $b = \beta$). Interestingly, this formula can be written as

$$p(\alpha\beta \mid ab) = p(\alpha\beta \mid a\beta) \cdot p(\alpha\beta \mid \alpha b),$$

a fact for which we have no obvious direct explanation.

(iii) Our results imply that $p(\alpha\beta \mid ab)$ "depends" only on λ_a and λ_b^*, (where λ^* denotes the partition conjugate to λ). More precisely, $p(\alpha\beta \mid ab) = (p(\alpha\beta \mid a'b')$ if $\lambda_a = \lambda_{a'}$ and $\lambda_b^* = \lambda_{b'}^*$, This shows that if one groups parts and conjugate parts of equal size then one has partitioned the Ferrers diagram of λ into "zones" of equal probability (of reaching $(\alpha\beta)$). For example, the following diagram shows the probabilities of reaching cell (4, 5) from all possible initial cells:

$$\begin{array}{ccccccc}
\frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{2} & \frac{1}{2} & 0 & 0 \\
\frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{2} & \frac{1}{2} & 0 & 0 \\
\frac{2}{5} & \frac{2}{5} & \frac{2}{5} & 1 & 1 & & \\
\frac{2}{5} & \frac{2}{5} & \frac{2}{5} & 1 & 1 & & \\
0 & 0 & 0 & & & & \\
0 & 0 & 0 & & & & \\
0 & 0 & 0 & & & &
\end{array}$$

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