Name: SOLUTIONS

Math 464/564 Fall 2017 Midterm Exam- 10/17/17

1. Below is the character table for a finite group:

	1	45	40	40	90	72	72
	C1	C2	C3	C4	C5	C6	C7
χ_1	1	1	1	1	1	1	1
χ_2	5	1	2	-1	-1	0	0
χ_3	5	1	-1	2	-1	0	0
χ_4	8	0	-1	-1	0	$(1-\sqrt{5})/2$	$(1+\sqrt{5})/2$
χ_5	8	0	-1	-1	0	$(1+\sqrt{5})/2$	$(1-\sqrt{5})/2$
χ_6	9	1	0	0	1	-1	-1
χ_7	10	-2	1	1	0	0	0
$\chi_7\cdot\chi_7$	100	4	1	1	0	0	0

a) Find the order of G.

b) Find the size of each conjugacy class $C1 - C7$. Write your answers above the table.

c) Is each element $g \in G$ conjugate to its inverse? Explain.

d) Write $\chi_7 \otimes \chi_7$ as a sum of irreducible characters.

e) Prove G is simple.

a. $|G| = 1^2+5^2+5^2+8^2+8^2+9^2+10^2 = 360$

b. Recall $C_i \cdot C_i = |C_G(g)| = |G|/|\text{class } g|$ $\quad C_1\cdot C_1 = 360 \quad C_2\cdot C_2 = 8 \quad C_3\cdot C_3 = 9$

$C_4\cdot C_4 = 9 \quad C_5\cdot C_5 = 4$

$C_6\cdot C_6 = 1+1+\frac{1}{4}(1-2\sqrt{5}+5)+\frac{1}{4}(1+2\sqrt{5}+5) = 1+1+12/4 = 5$

$C_7\cdot C_7 = 5$

c. Yes, since $\chi(g^{-1}) = \overline{\chi(g)}$ and all values are real

d. Using $\langle \chi_7\otimes\chi_7, \chi_i \rangle$ we obtain

$$\chi_7\otimes\chi_7 = \chi_1 + 2\chi_2 + 2\chi_3 + 2\chi_4 + 2\chi_5 + 3\chi_6 + 2\chi_7$$

e. Proof 1 If $1 < H \lhd G$ then G/H irreducibles give irreducibles of G w/ H in Kernel. However $\chi_2 - \chi_7$ all have trivial Kernel.

Proof 2 Normal subgroups are unions of conjugacy classes, including $\{e\}$. No such collection has order dividing 360.

2. Let G be a finite group acting on a set X and let $x \in X$. Prove the size of the orbit of x is equal to the index in G of the stabilizer of X. What does this theorem say about conjugacy class sizes?

Let $\mathcal{O}(x)$ and G_x be orbit & stabilizer. It suffices to find a bijection $\mathcal{O}(x) \to$ left cosets of G_x

Define $\psi: \mathcal{O}(x) \to G/G_x$ by $\psi(gx) = gG_x$ If

$\psi(g_1x) = \psi(g_2x)$ then $g_1G_x = g_2G_x \Rightarrow g_2^{-1}g_1 \in G_x \to g_2^{-1}g_1x = x \Rightarrow g_1x = g_2x$

so ψ is 1-1

Suppose $g_1x = g_2x$ Then $g_2^{-1}g_1 \in G_x \Rightarrow g_1G_x = g_2G_x \Rightarrow \psi(g_1x) = \psi(g_2x)$

so ψ is well-defined

ψ is clearly onto. //

When G acts on itself by conjugation we get

$$|\text{Conj class of } g| = [G : C_G(g)]$$

3.

a) Write down a basis $\{v_i\}$ for the Specht module $S^{(3,2)}$ and calculate the matrix of $\sigma = (1,2,3)$ in this basis.

b) Recall the S_n invariant bilinear form we defined on M^λ. Calculate the Gram matrix of this form on $S^{(3,2)}$. The Gram matrix is simply the matrix with $G_{ij} = \langle v_i, v_j \rangle$.

a. SYT of shape $(3,2)$: $t_1 = \begin{smallmatrix}123\\45\end{smallmatrix}$ $t_2 = \begin{smallmatrix}124\\35\end{smallmatrix}$ $t_3 = \begin{smallmatrix}125\\34\end{smallmatrix}$ $t_4 = \begin{smallmatrix}134\\25\end{smallmatrix}$ $t_5 = \begin{smallmatrix}135\\24\end{smallmatrix}$

$e_{t_1} = \overline{45} - \overline{15} - \overline{24} + \overline{12}$ $e_{t_2} = 35 - 15 - 23 + 12$ $e_{t_3} = 34 - 14 - 23 + 12$ $e_{t_4} = 25 - 15 - 23 + 13$

$e_{t_5} = 24 - 14 - 23 + 13$

$(123)e_{t_1} = 45 - 25 - 34 + 23 = e_{t_1} - e_{t_3} - e_{t_4} - e_{t_5}$

$(123)e_{t_2} = 15 - 25 - 13 + 23 = -e_{t_2}$

$(123)e_{t_3} = 14 - 24 - 13 + 23 = -e_{t_5}$

$(123)e_{t_4} = 35 - 25 - 13 + 12 = e_{t_2} - e_{t_4}$

$(123)e_{t_5} = 34 - 24 - 13 + 12 = e_{t_3} - e_{t_5}$

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & -1 & 0 \\ -1 & 0 & -1 & 0 & -1 \end{pmatrix}$$

b $$\begin{pmatrix} 4 & 2 & 1 & 1 & -1 \\ 2 & 4 & 2 & 2 & 1 \\ 1 & 2 & 4 & 1 & 2 \\ 1 & 2 & 1 & 4 & 2 \\ -1 & 1 & 2 & 2 & 4 \end{pmatrix}$$

4. Use Young's rule to decompose the permutation module $M^{(3,2,1)}$ into irreducible S_6 modules.

We need SSYT w/ content 1,1,1,2,2,3

$$\begin{matrix} 1\,1\,1 \\ 2\,2 \\ 3 \end{matrix} \qquad \begin{matrix} 1\,1\,1 \\ 2\,2\,3 \end{matrix} \qquad \begin{matrix} 1\,1\,1\,2 \\ 2 \\ 3 \end{matrix} \qquad \begin{matrix} 1\,1\,1\,2 \\ 2\,3 \end{matrix} \qquad \begin{matrix} 1\,1\,1\,2\,2 \\ 3 \end{matrix} \qquad 1\,1\,1\,2\,2\,3$$

$$\begin{matrix} 1\,1\,1\,3 \\ 2\,2 \end{matrix} \qquad \begin{matrix} 1\,1\,1\,2\,3 \\ 2 \end{matrix}$$

$$M^{321} \cong S^{321} \oplus S^{33} \oplus S^{411} \oplus 2\cdot S^{42} \oplus 2\cdot S^{51} \oplus S^{6}$$

5. Let $\sigma = (1,2,3,4)(5,6,7)(8,9,10)(11,12)(13,14)(15,16)$. Calculate the size of the centralizer of σ in S_{16}.

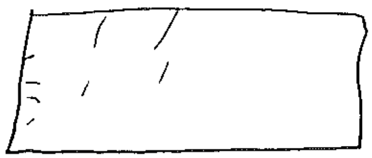
$$4 \cdot 3^2 \cdot 2! \cdot 2^3 \cdot 3! = 3456$$

6. For which n does S_n have a **nonlinear** irreducible character which remains irreducible upon restriction of S_{n-1}? Explain your answer.

Branching rule says $\chi^{\lambda}\downarrow_{S_{n-1}}$ is irreducible $\Leftrightarrow$

The Young diagram of λ has only one removable node

i.e. it is a rectangle

$\lambda =$ 

Now $\lambda = (n)$ and (1^n) are linear but as long as ~~λ~~ n is not prime we have others:

$$n = a \cdot b \rightarrow \lambda = (a, a, a, \ldots, a)$$

$\boxed{\text{Any } n > 2 \text{ not prime}}$

7. Construct the character table of S_5 in the following way:

a) Start with the trivial and sign character.

b) S_5 acts doubly transitively on $\{1,2,3,4,5\}$, which gives an irreducible character of degree 4. Tensoring with the sign gives another.

c) S_5 acts transitively on the 10 unordered 2 element subsets of $\{1,2,3,4,5\}$ giving a degree 10 permutation character. Use this and the characters already constructed to extract a 5-dimensional irreducible character. Tensoring with the sign gives another.

d) Complete the character table by any method you like.

We are searching for 7 irreducibles.

b. This gives irred char $\chi_3(\sigma) = |FP\sigma| - 1$.

Let $\chi_4 = \chi_3 \otimes \chi_2$.

c. This character ψ counts the # of unordered pairs fixed by σ. We put ψ below the table.

$\langle \psi, \chi_1 \rangle = 1 \quad \langle \psi, \chi_2 \rangle = 0 \quad \langle \psi, \chi_3 \rangle = 1 \quad \langle \psi, \chi_4 \rangle = 1$

Let $\chi_5 = \psi - \chi_1 - \chi_3$. Check $\langle \chi_5, \chi_5 \rangle = 1$ so χ_5 is irr.

Let $\chi_6 = \chi_5 \otimes \mathrm{sgn}$

d. χ_7 can be filled in easily with col $\perp$.

#elt	1	10	20	30	24	15	20
	e	(12)	(123)	(1234)	(12345)	$(12)(34)$	$(123)(45)$
χ_1	1	1	1	1	1	1	1
χ_2	1	-1	1	-1	1	1	-1
χ_3	4	2	1	0	-1	0	-1
χ_4	4	-2	1	0	-1	0	1
χ_5	5	1	-1	-1	0	1	1
χ_6	5	-1	-1	1	0	1	-1
χ_7	6	0	0	0	1	-2	0
ψ	10	4	1	0	0	2	1