Math 464/564 Fall 2017 Homework Number 6- Due 10/10/17

1. Sagan 2.12 #1

2. Sagan 2.12 # 4.

3. Sagan 2.12 #15

4. Sagan 2.12 # 16 Hint: For part (b) recall that the size of the conjugacy class is the index of the centralizer. So you will need to ask, for a given permutation σ, whether there are any odd permutations in its centralizer (following 15(b)).

5. Prove the character table of G has only real entries if and only if every element $g \in G$ is conjugate to its inverse.

6. Let χ be a faithful character of G.

a) Explain why, for each $i \geq 0$, the function χ^i defined by $\chi^i(g) = \chi(g)^i$ is also a character.

b) Suppose χ takes on r distinct values $\alpha_1, \alpha_2, \ldots, \alpha_r$. Define $G_i = \{g \in G \mid \chi(g) = \alpha_i\}$. Choose $\alpha_1 = \chi(1)$. Explain why $G_1 = \{e\}$.

c) Let ψ be an irreducible character of G. For $1 \leq i \leq r$ define $\beta_i = \sum_{g \in G_i} \overline{\psi(g)}$. Check that $\beta_1 \neq 0$ and show that for any $j \geq 0$:

$$\langle \chi^j, \psi \rangle = \frac{1}{|G|} \sum_{i=1}^{r} (\alpha_i)^j \beta_i.$$

d) Let A be the $r \times r$ matrix with ij entry $(\alpha_i)^{j-1}$ and let b be the row vector $b = (\beta_1, \beta_2, \ldots, \beta_r)$. Explain why A is invertible (Google Vandermonde determinant) and b is nonzero.

e) Using the the above, prove that $bA \neq 0$. Conclude that $\langle \chi^j, \psi \rangle \neq 0$ for some $0 \leq j \leq r-1$. You have now proved Burnside's theorem that, given a faithful character χ of G, every irreducible character is a constituent of some tensor power of χ.

f) Verify the theorem works for the degree four character χ_4 of S_5, as labelled on the character table of the last homework. How many tensor powers did you need to use?