Math 464/564 Fall 2017 Homework Number 5- Due 10/3/17

1. The character table for S_5 is below:

	e	(12)	$(12)(34)$	(123)	$(123)(45)$	(1234)	(12345)
χ_1	1	1	1	1	1	1	1
χ_2	1	-1	1	1	-1	-1	1
χ_3	4	-2	0	1	1	0	-1
χ_4	4	2	0	1	-1	0	-1
χ_5	5	-1	1	-1	-1	1	0
χ_6	5	1	1	-1	1	-1	0
χ_7	6	0	-2	0	0	0	1

a) Decompose the characters $\chi_5 \otimes \chi_6$ and $\chi_4 \otimes \chi_7$ into irreducible characters.

b) Choose χ one of the irreducible characters of S_4 of degree 3. Determine the decomposition of $\chi \uparrow^{S_5}$ using Frobenius reciprocity.

c) Let S_5 act on two element subsets of $\{1, 2, 3, 4, 5\}$, giving a 10-dimensional permutation module. Using the orthogonality relations, decompose this character into irreducibles.

2.

a) Write down the character table of the Klein 4-group V.

b) Consider the subgroup $\{e, (12)(34), (13)(24), (14)(23)\} \leq S_4$. Check it is isomorphic to V. Let ψ be a nontrivial irreducible character of V. Calculate the decomposition of $\psi \uparrow^{S_4}$.

3. Repeat the previous problem except replace V by the cyclic group of order 5 and the subgroup $\langle (12345) \rangle \leq S_5$.

4. Let G act on a set X and let χ be the corresponding permutation character. Recall that $\chi(g)$ counts the number of fixed points.

a) Let 1_G be the trivial character. Prove that $\langle \chi, 1_G \rangle$ is the number of orbits of G acting on X. Hint: Count the set $\{(g, x) \in G \times X \mid gx = x\}$ in two different ways, by first summing over g and then by summing over x.

b) Show that $\chi^2 := \chi \otimes \chi$ is the permutation character for the action of G on $X \times X$.

c) Explain why $\langle \chi, \chi \rangle = \langle \chi^2, 1_G \rangle$.

d) Suppose that G acts transitively on X (i.e. there is only one orbit). From above we know $\chi - 1_G$ is a character. Prove that $\chi - 1_G$ is irreducible if and only if G has precisely two orbits on $X \times X$.

e) Say the action is *doubly transitive* if for any pairs $(a, b), (c, d) \in X \times X$, with $a \neq b$ and $c \neq d$, there is a $g \in G$ so that $g(a, b) = (c, d)$. Prove this is equivalent to there being exactly two orbits on $X \times X$.

f) Prove that S_n acting on $\{1, 2, \ldots, n\}$ is doubly transitive. Conclude that S_n has an irreducible character of degree $n - 1$.