Math 464/564 Fall 2017 Homework Number 3- Due 9/19/17

1. Sagan 1.13 # 4.

2. Sagan 1.13 #7.

3. In this problem we give an alternate proof of Maschke's Theorem. So let V be a finite-dimensional G module with submodule U.

a) Show there is a linear map $\pi : V \to V$ such that $\mathrm{Image}(\pi) = U$, $\pi(u) = u$ for all $u \in U$, and $\pi \circ \pi = \pi$. The map π is called a *projection* onto U.

b) If π is as above, prove that $V \cong U \oplus \mathrm{kernel}(\pi)$ as vector spaces.

c) Define a new map as follows:

$$\tilde{\pi}(v) = \frac{1}{|G|} \sum_{g \in G} g\pi(g^{-1}v).$$

Prove that $\tilde{\pi}$ is also a projection onto U.

d) Verify that $\tilde{\pi}$ is a G-module map. Conclude that $V \cong U \oplus \ker(\tilde{\pi})$ as G-modules. Notice this proof makes quite clear why we only need that the characteristic of our field does not divide $|G|$, else we would be dividing by zero.

4. Recall that the *center* of a group is $Z(G) = \{z \in G \mid zg = gz \forall g \in G\}$. Let X be an irreducible representation of G and let $z \in Z(G)$. Prove that $X(z)$ is a multiple of the identity matrix.

5. Suppose U, V, W are G modules and let $T \in \mathrm{Hom}_G(U, V)$ and $S \in \mathrm{Hom}_G(V, W)$. Prove that $S \circ T \in \mathrm{Hom}_G(U, W)$. Conclude that $\mathrm{End}_G(V)$ is an algebra.

6. Let G be a finite group and let $\psi : G \to GL_2(\mathbb{C})$ be a representation. Suppose there are elements $g, h \in G$ so that the matrices $\psi(g)$ and $\psi(h)$ do not commute. Prove that ψ is irreducible. Would the same be true for $GL_3(\mathbb{C})$? Explain.