Lecture 9

Review An element a in an algebra A is <u>nilpotent</u> if $a^n = 0$ some $n > 0$.

<u>Situation</u> Considering subalgebras $L \leq gl(V)$, i.e. matrix (or linear) Lie algebras. So natural to consider adjoint action of L on itself via

$$ad\,X(Y) = [X, Y] \in gl(V)$$

<u>Def</u> $X \in L$ is <u>ad-nilpotent</u> if $ad\,X$ is nilpotent in $gl(L)$, i.e. the matrix for $ad\,X$ is nilpotent.

Equivalently $\exists n > 0$ so $\underbrace{[X, [X, [X, \ldots}_{n} [X, Y]]] \cdots] = 0 \ \forall Y$

<u>Remark</u>

1. If L is nilpotent then every $X \in L$ is ad-nilpotent.

<u>Proof</u> If $L^n = 0$ then <u>all</u> n-fold brackets are 0, not just one's with $n-1$ X's.

<u>Engel's Thm</u> Converse of above, not at all obvious.

<u>Lemma (Last time)</u>

Suppose $X \in L \leq gl(V)$. If X is nilpotent then it is ad-nilpotent.

i.e. $(A)^n = 0 \Rightarrow [A, [A, \ldots, [A, -] \cdots] = 0$

* Not vice-versa, e.g. I is ad nilpotent, clearly not nilpotent.

Summary

- nilpotent element in an algebra means $a^n = 0$, some n.
- For $x \in L$ a Lie algebra, x is ad-nilpotent if $\operatorname{ad} x$ is nilpotent in $\mathfrak{gl}(L)$

- Suppose $x \in \mathfrak{gl}(V)$, so $\operatorname{ad} x \in \mathfrak{gl}(\mathfrak{gl}(V))$ and x^n both make sense.

  Thm x nilpotent $\Longrightarrow$ ~~ad~~ x is ad-nilpotent.

  $$x^r = 0 \Rightarrow (\operatorname{ad} x)^{2r} = 0$$

  But not conversely, e.g. $x = Id$.

- Engel's Thm: L is nilpotent $\longleftrightarrow$ Every $x \in L$ is ad-nilpotent.

  $\longrightarrow$ easy

  $\longleftarrow$ hard

Weights and Weight Spaces

Setup V a vector space, $\mathfrak{gl}(V) = \{\text{linear maps } V \to V\}$, think of elements of $\mathfrak{gl}(V)$ as acting on V.

Problem Given a subalgebra $A \le \mathfrak{gl}(V)$, is there any 1-dimensional subspace $\langle v \rangle$ preserved by every $a \in A$, i.e.

* Is there $\vec{v} \neq 0$ simultaneously an eigenvector for every $a \in A$?

Def A a subalgebra of $\mathfrak{gl}(V)$. Say $0 \neq \vec{v} \in V$ is an eigenvector for A if $\exists\, \lambda : A \to F$ such that

$$a(\vec{v}) = \lambda(a)\vec{v} \qquad \forall a \in A.$$

Rmk $(a+b)\vec{v} = \lambda(a+b)\vec{v}$ by def

$\quad\; \parallel$

$a\vec{v} + b\vec{v} = (\lambda(a) + \lambda(b))\vec{v}$ So λ must be linear.

<u>Recall</u> $A^* = \{\lambda : A \to F \mid \lambda \text{ is linear}\}$ is <u>dual space</u> of linear functionals.

<u>Ex1</u>

Say A is the Lie subalgebra (1-dimensional abelian.) generated by a single matrix B. Then we are truly just considering eigenvectors of B.

<u>Ex2</u> $b(n,F) \leq gl(n,F)$, $\vec{v} = \vec{e}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$. Then

$$\begin{pmatrix} a_{11} & & * \\ & a_{22} & \\ 0 & & \ddots & a_{nn} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} = \begin{pmatrix} a_{11} \\ 0 \\ \vdots \\ 0 \end{pmatrix} \text{ so } \lambda\begin{pmatrix} a_{11} & & * \\ & \ddots & \\ 0 & & a_{nn} \end{pmatrix} = a_{11}$$

<u>Ex3</u> $d(n,F) \leq gl(n,F)$

$$\begin{pmatrix} \alpha_1 & & \\ & \alpha_2 & & 0 \\ 0 & & \ddots & \\ & & & \alpha_n \end{pmatrix} \cdot \vec{e}_i = \alpha_i e_i$$ so n different eigen-vectors

and $\lambda_i \begin{pmatrix} \alpha_1 & & 0 \\ & \alpha_2 & \\ 0 & & \ddots\ \alpha_n \end{pmatrix} = \alpha_i$.

<u>Ex4</u> $sl(n,F) \leq gl(n,F)$ No eigenvectors

<u>Def</u> Suppose $A \leq gl(V)$ and $\lambda : A \to F$ linear. Define the <u>λ-weight space</u>

$$V_\lambda = \{ v \in V \mid a(v) = \lambda(a) v \quad \forall a \in A \}$$

If $V_\lambda \neq 0$, say λ is a <u>weight</u> of A.

So weights are subsets of A^*.

Invariance Lemma

Suppose $L \subseteq gl(V)$ and A is an ideal of L. Suppose char $F=0$. Let $\lambda: A \to F$ be a weight. Then V_λ is L-invariant, i.e. every element of L maps V_λ to V_λ.

Special Case $\lambda = 0$ so let

$$W = V_0 = \{v \in V \mid a(v) = 0 \ \forall a \in A\}$$

Choose $w \in W$, $y \in L$. Want $y(w) \in W$, i.e. $a(y(w)) = 0 \ \forall a \in A$.

Trick $[a,y] = ay - ya$

Replace ay by $ya + [a,y]$

$$ay(w) = (ya + [a,y])(w)$$

$$= ya(w) + [a,y](w)$$
$$= 0 + 0 \quad \text{since } a \in A,\ [a,y] \in A.$$

General Case: Next time