Lecture 8

Review Given L, define two descending series

- $L^{(1)} = [L,L] \geq L^{(2)} \geq \cdots L^{(k)} = [L^{(k-1)}, L^{(k-1)}]$, derived series
- $L^{1} = [L,L] \geq L^{2} \cdots \geq L^{k} = [L, L^{k-1}]$ lower central series

Def L is <u>solvable</u> if $L^{(n)} = 0$ for some n, L is <u>nilpotent</u> if $L^{m} = 0$ some m

Thm L has a unique solvable ideal containing all solvable ideals, the <u>radical</u>, $\operatorname{rad} L$.

Def L is <u>semisimple</u> if $\operatorname{rad} L = 0$.

Rmk $L/\operatorname{rad} L$ is semisimple

Exercise ~~Prove L/J is semisimple iff~~

Goal
- Understand solvable Lie algebras.
- Classify all simple Lie algebras $/\mathbb{C}$.

Linear Algebra Review

V a vector space $/F$, $T: V \to V$ linear map. Say $\lambda \in F$ is an <u>eigenvalue</u> of T if $\exists\, \vec{v} \in V$ so $\vec{v} \neq 0$ and $T(\vec{v}) = \lambda \vec{v}$. Then $\vec{v}$ is called an <u>eigenvector</u>.

Exercise The set of eigenvectors of eigenvalue λ, together with 0, forms a subspace of V, the λ-eigenspace

$*$ λ-eigenspace $=$ Nullspace $(T - \lambda\, Id)$

Similarly for matrices a column vector $\vec{u} \neq 0$ is an e-vector of $A \in gl(n, F)$ if $A\vec{u} = \lambda\vec{u}$.

<u>Finding eigenvalues</u> $\lambda \in F$ is an eigenvalue $\leftrightarrow A\vec{u} = \lambda\vec{u}$ some $\vec{u} \neq 0$

$\leftrightarrow (A - \lambda I)\vec{u} = 0$ some $\vec{u} \neq 0$

$\leftrightarrow$ Nullspace $A - \lambda I \neq 0$

$\leftrightarrow \det(A - \lambda I) = 0$

$\leftrightarrow \lambda$ is a root of $\underbrace{\det(A - xI)}_{\text{characteristic polynomial}}$

<u>Finding eigenspaces</u>

Given λ an eigenvalue, the corresponding eigenspace is Nullspace $(A - \lambda I)$

<u>Ex</u> $A = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 2 \\ 0 & 0 & 5 \end{pmatrix}$ c.p. $= \det\begin{pmatrix} 1-x & 0 & 0 \\ -1 & 1-x & 2 \\ 0 & 0 & 5-x \end{pmatrix} = -(x-1)^2(x-5)$

$\lambda = 1$ eigenspace:

$A - I\mathrm{d} = \begin{pmatrix} 0 & 0 & 0 \\ -1 & 0 & 2 \\ 0 & 0 & 4 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$ $x = 0$ $z = 0$

Basis $\{(0,1,0)\}$

$\lambda = 5$ $A - 5I\mathrm{d} = \begin{pmatrix} -4 & 0 & 0 \\ -1 & -4 & 2 \\ 0 & 0 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 0 \\ 0 & -4 & 2 \\ 0 & 0 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -1/2 \\ 0 & 0 & 0 \end{pmatrix}$ $x = 0$, $y = \frac{1}{2}z$

Basis $\{(0,1,2)\}$

<u>Def</u> A is <u>diagonalizable</u> if there is a basis of eigenvectors

$\leftrightarrow$ sum of dimensions of e-spaces $= n$

$\leftrightarrow \exists P$ invertible w/ $PAP^{-1} = \begin{pmatrix} \lambda_1 & & 0 \\ & \lambda_2 & \\ 0 & & \ddots \ \lambda_n \end{pmatrix}$

<u>Ex</u> Suppose A has n-distinct eigenvalues in F. Then A is diagonalizable

<u>Use:</u> Eigenvectors w/ distinct eigenvalues are lin. ind.

Exercise 1.17

Let V be n-dim. vector space over $\mathbb{C}$. Let $L = gl(V)$.
Suppose $x \in gl(V)$ is diagonalizable w/ e-values $\lambda_1, \lambda_2, \dots, \lambda_n \in \mathbb{C}$
Show $ad\, x \in gl(L)$ is diagonalizable w/ evalues $\{\lambda_i - \lambda_j\}$.

Proof Choose a basis $\{v_1, v_2, \dots, v_n\}$ so $x(v_i) = \lambda_i v_i$.

Choose a basis of $L = gl(V)$ by $T_{ij} : \begin{matrix} v_j \to v_i \\ \text{rest} \to 0 \end{matrix}$ (corr to matrix unit e_{ij})

$$
\begin{aligned}
(ad\, x)(T_{ij})(v_k) &= (x \circ T_{ij} - T_{ij} \circ x)(v_k) \\
&= x(T_{ij}(v_k)) - T_{ij}(x(v_k)) \\
&= x(\delta_k^j v_i) - T_{ij}(\lambda_k v_k) \\
&= \delta_k^j \lambda_i v_i - \delta_k^j \lambda_k v_i \\
&= \delta_k^j (\lambda_i - \lambda_k) v_i
\end{aligned}
$$

Thus $(ad\, x)(T_{ij}) = (\lambda_i - \lambda_k) T_{ij}$ //

CHPT 5 Subalgebras of $gl(V)$

Remark Let L be a subalgebra of $gl(V)$, $gl(V)$ has bracket but also composition, but L may not be closed under composition

Def An element a in an algebra A is nilpotent if $a^n = 0$ for some n

Rmk Nilpotent matrices have 0 as only e-value, c.p. $= x^n$
Opposite of diagonalisable

Lemma Let $L \subseteq gl(V)$. Suppose $x: V \to V$, so $x \in L$.
If x is nilpotent, then $\operatorname{ad} x : L \to L$ is nilpotent.

Proof Suppose $y \in L$. $(\operatorname{ad} x)(y) = [x,y] = x \circ y - y \circ x$

$$(\operatorname{ad} x)^2(y) = (\operatorname{ad} x)(\operatorname{ad} x(y)) = x \circ (x \circ y - y \circ x) - (x \circ y - y \circ x) \circ x$$
$$= x \circ x \circ y - x \circ y \circ x - x \circ y \circ x + y \circ x \circ x$$

Consider
$$(\operatorname{ad} x)^{2r}(y) = \ldots \quad \text{all} \quad x^i \circ y \circ x^{2r-i}, \text{ so all } 0. \ //$$

Def Suppose A is a subalgebra of $gl(V)$. Say $\vec{v} \in V$ is an <u>eigenvector for A</u> if it is an eigenvector of <u>every</u> $a \in A$ (perhaps w/ different eigenvalues).

$$\text{i.e. } a(v) \in \operatorname{span}\langle v \rangle \quad \forall a \in A.$$

Ex $A = b(n,F)$ $\quad e_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ is an eigenvector of A.

$A = d(n,F)$ diagonal matrices. Each e_i is an eigenvector

Rmk Suppose $\vec{v}$ is an eigenvector for A. Then
$$a(v) = \lambda(a) v \quad \text{gives a function}$$
$$\lambda : A \to F.$$

Def Given a <u>function</u> $\lambda : A \to F$, the corresponding eigenspace is
$$V_\lambda := \{ v \in V \mid a(v) = \lambda(a) v \quad \forall a \in A \}$$

Rmk Suppose $\lambda : A \to F$ and $0 \neq v \in A_\lambda$. Let $a, b \in A$ $\alpha, \beta \in F$.
Then

$$(\alpha a + \beta b)(\vec{v}) = \alpha a(v) + \beta b(v)$$

$$= \cancel{\alpha\lambda v + \beta}\ \alpha\lambda(a)v + \beta\lambda(b)v$$

$$= (\alpha\lambda(a) + \beta\lambda(b))v$$

$$= \lambda(\alpha a + \beta b)\vec{v}$$

Thus $\lambda : A \to F$ cannot have nonzero eigenspace
unless λ is linear, i.e.
$\lambda \in A^*$.