Lecture 7

Review

Derived series: $L^{(1)} = L'$, $L^{(2)} = [L^{(1)}, L^{(1)}]$, .., $L^{(k)} = [L^{(k-1)}, L^{(k-1)}]$

Def L is solvable if $L^{(m)} = 0$ for some $m \geq 1$

If $L \supset L^{(1)} \supset \dots \supset L^{(m-1)} \supset L^{(m)} = 0$ then each $L^{(k)}/L^{(k+1)}$ is abelian and no other series descends as fast with abelian quotient.

Ex $\mathfrak{b}(n, F)$ is solvable.
$\mathfrak{sl}(n, \mathbb{C})' = \mathfrak{sl}(n, \mathbb{C})$ so not solvable, $n \geq 2$

Recall

Prop 1 $\varphi: L_1 \to L_2$ a surjection. Then $\varphi(L_1^{(k)}) = L_2^{(k)}$ $\forall k \geq 1$.

Prop 2 Suppose L a Lie algebra.

a. If L is solvable, so is every subalgebra and homomorphic image.

b. Let J be an ideal. If J and L/J are solvable, then L is solvable.

c. Suppose I, J are solvable ideals. Then so is $I+J$.

Proof

a. Hom. image follows by Prop 1. Suppose $K \subset L_1$ a subalgebra. Clearly $K^{(m)} \subseteq L_1^{(m)}$ so K is solvable.

b. Let $\pi: L \to L/J$. By Prop 1, $(L/J)^{(k)} = \pi(L^{(k)}) = L^{(k)}+J/J$.
Suppose $(L/J)^{(m)} = 0$, then $L^{(m)} \subseteq J$.
Suppose $(J)^{(s)} = 0$. Then $L^{(m+s)} = 0$.

c. $I+J/I \cong J/I \cap J$ so $I+J/I$ and I are solvable $\Rightarrow$ $I+J$ is.

<u>Cor/Def</u> Suppose L is fin dim. Then L contains a unique solvable ideal which contains all solvable ideals. It is called the <u>radical</u> of L, rad L.

<u>Proof</u> Choose a solvable ideal R of maximal possible dimension. Suppose I is a solvable ideal. So is $R+I$, but $\dim(R+I) \geq \dim R$, with equality $\Leftrightarrow I \subseteq R$ //

<u>Key Def</u> L is <u>semisimple</u> if rad $L = 0$. (i.e. no solvable ideals)

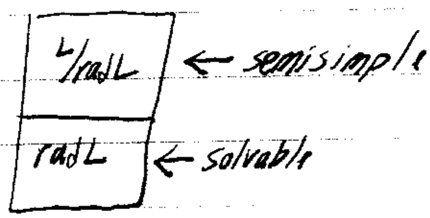
<u>Ex</u> 1. $\text{rad}(L) = L \Leftrightarrow L$ is solvable.

2. $\mathfrak{sl}_2(\mathbb{C})$ is semisimple.

3. $L/\text{rad}\,L$ is always semisimple!

<u>Proof</u> Suppose not. Let $\bar{J}$ be a nonzero solvable ideal. Then $\bar{J}$ corresponds to $J \supset \text{rad}\,L$, with $J/\text{rad}\,L \cong \bar{J}$. Thus J is solv. $\divideontimes$

<u>Rmk</u>



Goal: Understand both pieces

<u>Def</u> L is <u>simple</u> if nonabelian and no nontrivial ideals.

Simple $\Rightarrow$ Semisimple

<u>Later</u> Every semisimple Lie algebra is a $\oplus$ of simple Lie alg's.

<u>Later</u> $F = \mathbb{C}$

1. Lie's Thm: Solvables $\rightsquigarrow$ subalgebras of $\mathfrak{b}(n, \mathbb{C})$

2. Major Thm: Classify all simple Lie algebras over $\mathbb{C}$

Nilpotent Lie Algebras

Def $L^1 = L'$; $L^2 = [L, L']$ .. $L^k = [L, L^{k-1}]$

Then $L \supseteq L' \supseteq L^2 \supseteq ..$ is the <u>lower central series</u>

Rmks

1. Clearly $L^{(k)} \subseteq L^k$, and each L^k is an ideal in L.

2. $L^k / L^{k+1} \subseteq Z(L/L^{k+1})$

   $x \in L^k$, $[x + L^{k+1}, y + L^{k+1}] = [x,y] + L^{k+1} = 0$.

Def L is <u>nilpotent</u> if $L^m = 0$ for some $m \geq 0$.

Clearly nilpotent $\Rightarrow$ solvable.

Ex 1. $b(n,F)$ is solvable but not nilpotent, also 2-dim nonabelian.

2. $n(n,F)$ is nilpotent.

Lemma

a. L nilpotent $\Rightarrow$ any subalgebra is nilpotent, any quotient is nilpotent.

b. $L/Z(L)$ nilpotent $\Rightarrow$ L nilpotent.

c. L nilpotent $\Rightarrow Z(L) \neq 0$

Pf a. clear, $K \subseteq L \Rightarrow K^s \subseteq L^s$

b. Check $(L/Z(L))^k = L^k + Z(L) / Z(L)$

so if $(L/Z(L))^k = 0$ then $L^k \subseteq Z(L)$ so $L^{k+1} = 0$.

c. Last term $\neq 0$ is central //

(**) L/I, I both nilpotent $\not\Rightarrow$ L is nilpotent