HW: 4.3, 4.4, 4.5, 4.6
2.10, 3.1

Lecture 6

Quotient Space Review $W \subseteq V$ a subspace. A coset is $v + W = \{v + w \mid w \in W\}$

Rmk $v + W = v + \tilde{w} + (W - \tilde{w})$ so if $\tilde{w} \in W$ then $v + W = v + \tilde{w} + W$.
Conversely if $v_1 + W = v_2 + W$ then $v_1 + 0 = v_2 + w \Rightarrow v_1 - v_2 \in W$.

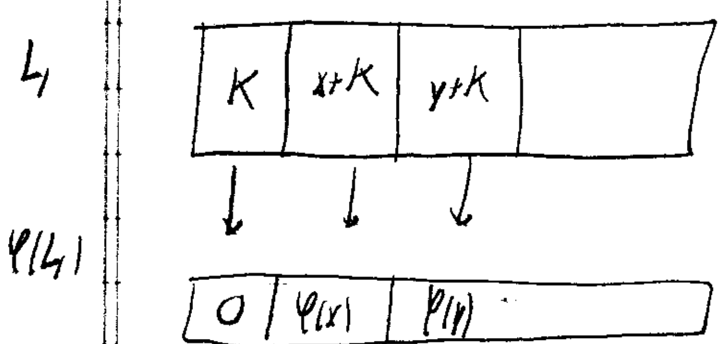
Conclude $v_1 + W = v_2 + W \leftrightarrow v_1 - v_2 \in W$.

Rmk $v_1 \sim v_2 \leftrightarrow v_1 - v_2 \in W$ is an equivalence relation, i.e. distinct cosets are disjoint.

Recall $\varphi : L_1 \to L_2$ Lie alg. homom. Then $L_1/\ker\varphi \cong \operatorname{Im}\varphi$.

$$x + K = y + K \leftrightarrow x - y \in K \leftrightarrow \varphi(x-y) = 0 \leftrightarrow \varphi(x) = \varphi(y)$$

$$* \quad x + K = \varphi^{-1}\{\varphi(x)\} \quad *$$



Each coset has all its elements mapped to a single element in L_2.

Example $\operatorname{ad} : L \to gl(L)$, $\operatorname{Im}(\operatorname{ad}) = \operatorname{IDer}(L) \subseteq \operatorname{Der}(L) \subseteq gl(L)$

$L/Z(L) \cong \operatorname{IDer} L \qquad x + Z = y + Z \leftrightarrow x - y \in Z(L)$
$\leftrightarrow [x-y, w] = 0 \quad \forall w \in L$
$\leftrightarrow [x, w] = [y, w] \quad \forall w$

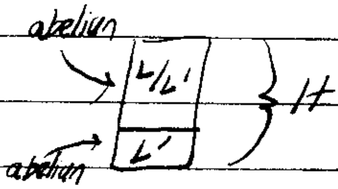
i.e. x, y in same coset of $Z(L)$ if $[x, -]$, $[y, -]$ do the same thing!

Solvable Lie Algebras

Recall 1. $L' = [L,L] = \text{span}\{[x,y] \mid x,y \in L\}$ is the derived algebra.
2. L/I is abelian $\Leftrightarrow L' \subseteq I$.

Question: Given L, is it "built up" out of abelian Lie algebra slices?

Ex H = Heisenberg $L' = Z(L)$ is 1-dim



If L' is not abelian, try $(L')'$, etc...

Def Let $L^{(1)} = L' = [L,L]$, $L^{(2)} = [L^{(1)}, L^{(1)}]$, ... $L^{(n)} = [L^{(n-1)}, L^{(n-1)}]$

1. $L \supseteq L^{(1)} \supseteq L^{(2)} \supseteq \cdots$ is called the derived series of L.
2. $L^{(i)}/L^{(i+1)}$ is abelian
3. Each $L^{(i)}$ is an ideal in L, not just in $L^{(i-1)}$.

Def L is solvable if $L^{(m)} = 0$ for some $m \geq 1$.

Rmk If $L^{(i)} = L^{(i+1)}$ then $L^{(i)} = L^{(n)}$ $\forall n \geq i$.

Ex. H is solvable, $L^{(2)} = 0$.

Important Example $L = \mathfrak{b}(n,F) = \{\text{upper } \Delta \text{ matrices in } \mathfrak{gl}(n,F)\}$ is solvable

- $L' = \mathfrak{n}(n,F) = \left\{\begin{pmatrix} 0 & & * \\ & \ddots & \\ 0 & & 0 \end{pmatrix}\right\}$

- $L^{(2)} = \left\{\begin{pmatrix} 0 & 0 & & * \\ & 0 & 0 & \\ & & \ddots & \ddots \\ 0 & & & 0 \end{pmatrix}\right\}$ ... $L^{(m)} = 0$

Ex $L = \mathfrak{sl}(n,\mathbb{C})$ $L' = \mathfrak{sl}(n,\mathbb{C})$ Thus not solvable.

<u>Proof</u> Must express each basis elt as a linear comb of $[\,,\,]$

<u>Ex</u> $\mathfrak{sl}(2,\mathbb{C})$ $[e,f]=h$, $[h,e]=2e$, $[h,f]=-2f$

Rmk If L is solvable, it is built out of abelian slices

Lemma Suppose L has ideals $L = I_0 \supseteq I_1 \supseteq \dots \supseteq I_m = 0$ with I_{k-1}/I_k abelian for $1 \le k \le m$. Then L is solvable.

<u>Proof</u> We prove by induction that $L^{(k)} \subseteq I_k$ and thus $L^{(m)} = 0$.

<u>Base Case</u> $I_0/I_1 = L/I_1$ is abelian so $L^{(1)} \subseteq I_1$ by previous result.

<u>Inductive Step</u> Assume $L^{(k-1)} \subseteq I_{k-1}$ for some $k \ge 2$.

By assumption I_{k-1}/I_k is abelian so $[I_{k-1}, I_{k-1}] \subseteq I_k$.
But $L^{(k-1)} \subseteq I_{k-1}$ so $\underbrace{[L^{(k-1)}, L^{(k-1)}]}_{= L^{(k)}} \subseteq [I_{k-1}, I_{k-1}] \subseteq I_k$. //

Rmk The Lemma shows the derived series is the fastest descending among all series with abelian quotients

Prop (Generalizes HW) Suppose $\varphi: L_1 \to L_2$ is onto. Then $\varphi(L_1^{(k)}) = L_2^{(k)}$

<u>Proof</u> $\varphi([x,y]) = [\varphi(x), \varphi(y)]$ so $\varphi(L_1^{(1)}) \subseteq L_2^{(1)}$.
But let $\varphi(x), \varphi(y)$ be arbitrary elements of L_2 (of this form since φ is onto)
Then $[\varphi(x), \varphi(y)] = \varphi([x,y])$ so $L_2^{(1)} \subseteq \varphi(L_1^{(1)})$

Thus $\varphi(L_1^{(1)}) = L_2^{(1)}$.

Now proceed by induction: Suppose $\varphi(L_1^{(k-1)}) = \varphi(L_2^{(k-1)})$.

Elements of $L_1^{(k)}$ are linear combinations of $[x,y]$ with $x,y \in L_1^{(k-1)}$.
But then

$$\varphi([x,y]) = [\varphi(x), \varphi(y)] \in [L_2^{(k-1)}, L_2^{(k-1)}] \text{ by assumption}$$
$$= L_2^{(k)}$$

Thus $\varphi(L_1^{(k)}) \subseteq L_2^{(k)}$

Now choose $[u,v] \in L_2$ with $u,v \in L_2^{(k-1)}$, so $[u,v] \in L_2^{(k)}$.

By inductive assumption $u = \varphi(x)$, $v = \varphi(y)$ some $x,y \in L_1^{(k-1)}$.

Thus $[u,v] = \varphi([x,y]) \in \varphi(L_1^{(k)})$ so $L_2^{(k)} \subseteq \varphi(L_1^{(k)})$

Thus $L_2^{(k)} = \varphi(L_1^{(k)})$ //

Thm Let L be a Lie algebra.

(a) If L is solvable, so is every subalgebra and every homomorphic image of L (i.e. $\varphi(L)$ for φ a homomorphism)

(b) Suppose $I \subseteq L$ an ideal, with I and L/I solvable. Then L is solvable.

(c) If ~~[illegible]~~ I, J are solvable ideals, so is $I + J$.

Proof

(a) $L_1 \subset L$ subalg, obviously $L_1^{(k)} \subseteq L^{(k)}$ ✓

Since $L^{(m)}=0 \Rightarrow \varphi(L)^{(m)}=0$, follows by Prop

(b) $\pi : L \to L/I$, use Prop to get

$$(L/I)^{(k)} = L^{(k)}+I/I$$

Since L/I is solvable, $(L/I)^{(m)}=0$ some m

$$\Rightarrow L^{(m)} \subseteq I.$$

Now $I^{(s)}=0$ some s $\Rightarrow$ $L^{(m+s)}=0$

(c) $I+J/I \cong J/I\cap J$ is solvable by above

I solv   use (b).