Lecture 5

<u>Review</u> $J \subseteq L$ an ideal, L/J quotient Lie algebra via $[x+J, y+J] = [x,y] + J$.

<u>Thm</u> Let $\varphi: L_1 \to L_2$ be a Lie alg homom. Then $L_1/\ker\varphi \cong \varphi(L_1)$ as Lie alg.

<u>Rmk</u> There is a natural quotient map $\pi: L \to L/J$ given by $\pi(x) = x + J$ (aka $\bar{x}$), $\ker\pi = J$.
The theorem says every homomorphism is in effect just a quotient map.

<u>Arithmetic With Ideals</u>

Let $I, J \subseteq L$ be ideals.

- $I \cap J$ is clearly an ideal
- $I + J = \{i + j \mid i \in I, j \in J\} = \text{span}\langle I, J\rangle$ also clearly an ideal.

<u>Def</u> $[I,J] = \text{span}\{[i,j] \mid i \in I, j \in J\}$

<u>Prop</u> $[I,J]$ is an ideal

<u>Proof</u> Let $x \in L$, $i \in I$, $j \in J$. $[x,[i,j]] = -[i,[j,x]] - [j,[x,i]]$
$= -[i,\tilde{j}] + [\tilde{i},j] \in I + J$ //

<u>Rmk</u> Need span, $[i,j] + [\tilde{i},\tilde{j}]$ not necessarily of the form $[\tilde{i},\tilde{j}]$.

<u>Example</u> $I = J = L$, then $L' = [L,L]$ is called <u>the derived algebra</u>.

<u>Exercise</u> Let $J \subseteq L$ an ideal. Then L/J is abelian if and only if $L' \subseteq J$.

<u>Pf</u> L/J abelian $\leftrightarrow [x+J, y+J] = 0 \ \forall x,y$
$\leftrightarrow [x,y] + J = 0 \ \forall x,y$
$\leftrightarrow [x,y] \in J \ \forall x,y \leftrightarrow L' \subseteq J$ //

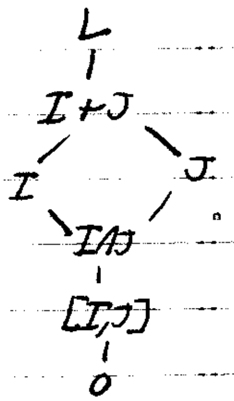
Some more $\cong$ Theorems

Thm Suppose $I \subseteq J \subseteq L$ are ideals. Then $L/J \cong \frac{L/I}{J/I}$

Proof Define $\varphi : L/I \to L/J$ by $\varphi(x+I) = x+J$.
Check well-defined, onto, kernel $= J/I$. Use previous Thm. //

Thm Let $I, J \subseteq L$. Then $I+J/J \cong I/I \cap J$

Proof Define φ by $I \overset{i}{\hookrightarrow} I+J \xrightarrow{\pi} I+J/J$
$\varphi(i) = i+J$.



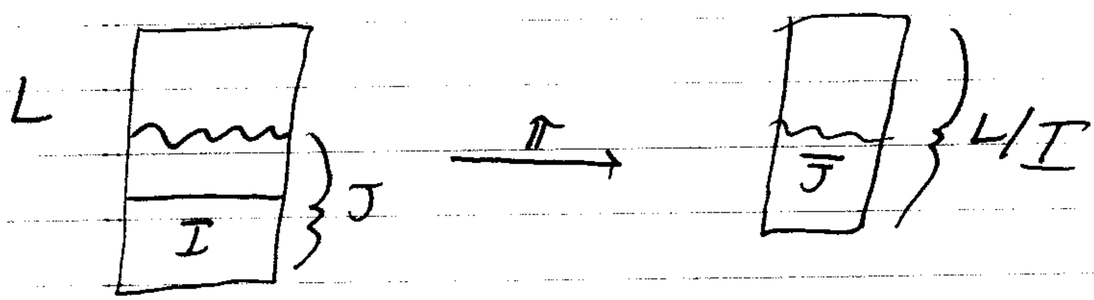
Since $i+j+J = i+J$, φ is onto. $\mathrm{Ker}\,\varphi = J \cap I$ so
$I/I \cap J \cong I+J/J$.

Thm Suppose $I \subseteq L$ an ideal. There is a bijective correspondence between

Ideals of L/I $\longleftrightarrow$ Ideals of L containing I

Pf $\longrightarrow$ Let $\bar{J} \subseteq L/I$. Define $J = \{x \in L \mid x+I \in \bar{J}\}$

$\longleftarrow$ Let $I \subseteq J \subseteq L$. Let $\bar{J} = \{j+I\} \subseteq L/I$.



Low-Dimensional Nonabelian Lie Algs

<u>$\dim L = 1$</u> $L = \langle x \rangle$ but $[x,x]=0$ so L must be abelian.

<u>$\dim L = 2$</u> Let $L = \langle x, y \rangle$. If $[x,y]=0$, L is abelian. If not then L' is 1-dim.
Choose a basis $L = \langle u, v \rangle$ with $L' = \langle u \rangle$
Then $[u,v] = \alpha u$ some $\alpha \neq 0$. Let $\tilde{v} = \frac{1}{\alpha} v$. Then

$$L = \langle u, \tilde{v} \rangle \text{ with } [u,v] = u.$$

must check J.I. to see this actually gives a Lie alg

<u>Conclude</u>: If $\dim L = 2$, L is abelian or $L = \langle u, v \rangle$ w/ $[u,v] = u$.

<u>$\dim L = 3$</u> Classify based on $\dim L'$ (1,2,3) and $Z(L)$ (0,1,2)

<u>Case 1</u> $\dim L' = 1$, $L' \subseteq Z(L)$

Choose f,g w/ $[f,g] \neq 0$ so $L' = \langle [f,g] \rangle$. Let $z = [f,g]$
so $z \in Z(L)$. If $z = af + ag$ then $0 = [z,f] = b[g,f] \to b = 0$
~~If~~ $0 = [z,g] = a[f,g] \to a = 0$.

Thus $\{f, g, z\}$ is a basis

$$L \cong \langle f, g, z \mid [f,g] = z, \quad [f,z] = [g,z] = 0 \rangle$$

Called Heisenberg Lie Algebra $\cong$ 3x3 strictly upper Δ

$$f \mapsto \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad g \mapsto \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \quad h \mapsto \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

<u>Rmk</u> $L' = Z(L)$

Case 2 $\dim L' = 1$, $L' \subseteq Z(L)$

Answer: $L \cong \underset{\text{nonab}}{2\dim} \oplus 1\dim = \langle x, y, z \rangle$ $\quad [x,y] = x$, $[x,z] = [y,z] = 0$

Case 3 $\dim L' = 2$, so Let $L' = \langle y, z \rangle$, $L = \langle x, y, z \rangle$, over $\mathbb{C}$.

Lemma (a) L' is abelian

(b) $\operatorname{ad} x : L' \to L'$ is an $\cong$

Pf Let $[y,z] = \alpha y + \beta z$, consider $\operatorname{ad} y$ acting on $\langle x, y, z \rangle$

$$\operatorname{ad} y \rightsquigarrow \begin{pmatrix} 0 & 0 & 0 \\ * & 0 & \alpha \\ * & 0 & \beta \end{pmatrix}$$

Now trace $(\operatorname{ad} y) = 0$ since $y \in L' \Rightarrow \beta = 0$

Repeat for $\operatorname{ad} z$ to get $\alpha = 0$

Thus $[y,z] = 0$ so L' is abelian

(b) Since $[y,z] = 0$, $L' = \operatorname{span}\langle [x,y], [x,z] \rangle$, but $\dim L' = 2$ so $[x,y]$, $[x,z]$ are lin ind //

Subcase (i) Suppose $\exists\, x \notin L'$ with $\operatorname{ad} x : L' \to L'$ diagonalisable

Assume y, z are eigenvectors

$[x,y] = \lambda y$, replace by $\tilde{x} = \frac{1}{\lambda} x$ so $[\tilde{x}, y] = y$

Thus $\operatorname{ad} \tilde{x}$ has matrix $\begin{pmatrix} 1 & 0 \\ 0 & \mu \end{pmatrix}$ $\mu \in \mathbb{C}^*$

$$L_\mu = \langle x, y, z \rangle \quad [x,y] = y \quad [x,z] = \mu z \quad [y,z] = 0$$

Check L_μ is a Lie algebra, $L_\mu \cong L_\tau \Leftrightarrow \mu = \tau$ or $1/\tau$

<u>Subcase (ii)</u> No $x \notin L'$ are diag. Choose $x \in L'$. Chose an eigenvector $y \in L'$ for ad x. WLOG $[x,y] = y$.
Extend to basis $\{y, z\}$ of L'.

$$[x,z] = \lambda y + \mu z, \quad \text{WLOG } \lambda = 1$$

by scaling z

$$\operatorname{ad} x \begin{pmatrix} 1 & 1 \\ 0 & \mu \end{pmatrix} \Rightarrow \mu = 1$$

$$L = \langle x, y, z \rangle \qquad \begin{aligned} [x,y] &= y \\ [x,z] &= y + \cancel{\mu} z \\ [y,z] &= 0 \end{aligned}$$

Just one algebra h

<u>Case 4</u> $L' = L$

Only $\mathfrak{sl}_2(\mathbb{C})$