Lecture 3

Review *Lie algebra* is a vector space L together with a bilinear map $[,]: L \times L \to L$ such that:

(L1): $[x,x]=0$, (L2) (Jacobi id) $[x[y,z]]+[y[z,x]]+[z[x,y]]=0$ $\forall x,y,z \in L$.

Rmk (L1) $\Rightarrow$ (L1)': $[x,y] = -[y,x]$

Def A *Lie subalgebra* is a subspace K such that:

$$[K,K] := \{[x,y] \mid x,y \in K\} \subseteq K.$$

- many of our examples were subalgebras of $\mathfrak{gl}(n,F)$.

Def An *ideal* is a subspace K such that:

$$[K,L] := \{[k,x] \mid k \in K, x \in L\} \subseteq K$$

Rmk ideal $\Rightarrow$ subalgebra

Key example

The *center* $Z(L) := \{z \in L \mid [x,z]=0 \ \forall x \in L\}$ is trivially an ideal.

$$Z(L)=L \leftrightarrow L \text{ is abelian}$$

Rmk $[x,z]=0 \ \forall x \in L \leftrightarrow \{[x_i,z]=0 \mid x_i \text{ in a basis of } L\}$ so given structure constants, finding $Z(L)$ is just solving a system of linear equations

Ex $L=\mathfrak{sl}_2(F)$, $e=\begin{pmatrix}0&1\\0&0\end{pmatrix}$, $f=\begin{pmatrix}0&0\\1&0\end{pmatrix}$, $h=\begin{pmatrix}1&0\\0&-1\end{pmatrix}$ $\quad [h,e]=2e$, $[h,f]=-2f$, $[e,f]=h$

<u>Homomorphisms</u> General setup: $(X, *)$, $(Y, \star)$ sets with binary ops $*, \star$, $T: X \to Y$

$*$	x_2
x_1	$x_1 * x_2$

$\star$	$T(x_2)$
$T(x_1)$	$T(x_1) \star T(x_2)$

$$\text{Want: } T(x_1 * x_2) = T(x_1) \star T(x_2)$$

<u>Ex</u> Linear map: $T(x_1 + x_2) = T(x_1) + T(x_2)$, $T(\lambda x_1) = \lambda T(x_1)$

Group homom, Ring homom, etc..

<u>Def</u> Let L_1, L_2 be Lie algebras over F. A map $\varphi: L_1 \to L_2$ is a <u>Lie algebra homomorphism</u> if it is linear and:

$$\varphi([x,y]) = [\varphi(x), \varphi(y)] \quad \forall x, y \in L_1$$

If φ is 1-1 and onto then φ is an <u>isomorphism</u> and $L_1 \cong L_2$.

<u>Ex</u> (Most important of all!)

L a Lie alg. Recall $\mathfrak{gl}(L)$ = Lie alg of all linear maps $L \to L$,
$[a,b] = a \circ b - b \circ a$.

The <u>adjoint homomorphism</u>

$$ad: L \longrightarrow \mathfrak{gl}(L) \quad \text{is defined by}$$

$$(ad\, x)(y) = [x,y] \quad \forall x, y \in L, \text{ i.e. } x \longrightarrow [x, -]$$

- $[x, -]$ clearly linear

<u>Need</u> $ad([x,y]) = [ad\, x, ad\, y]$, so apply both sides to $z \in L$.

Need $[[X,Y],Z] = \mathrm{ad}X \circ \mathrm{ad}Y(Z) - (\mathrm{ad}Y \circ \mathrm{ad}X)(Z)$
$$= [X,[Y,Z]] - [Y,[X,Z]]$$
use $[a,b] = -[b,a]$ + J.I. //

Lemma Let $\varphi: L_1 \to L_2$ be a Lie alg. homo. Then $\mathrm{Ker}\,\varphi = \{x \mid \varphi(x) = 0\}$ is an ideal.

Proof Let $x \in \mathrm{Ker}\,\varphi$, $y \in L_1$.
$$\varphi([x,y]) = [\varphi(x), \varphi(y)] = [0, \varphi(y)] = 0$$
so $[x,y] \in \mathrm{Ker}\,\varphi$ //

Ex $\mathrm{Ker}(\mathrm{ad}) = Z(L)$ by definition.

Rmk ad is our first example of a representation, namely a Lie alg homo. $L \to \mathfrak{gl}(V)$ for some V.

Derivations

Let A be an F-algebra. A derivation of A is a linear map $D: A \to A$ such that
$$D(ab) = aD(b) + D(a)b \quad \forall a,b \in A.$$

Prop Let $\mathrm{Der}\,A$ be the set of all derivations of A. Then $\mathrm{Der}\,A$ is a Lie subalgebra of $\mathfrak{gl}(A)$.

Proof

1. Check $\mathrm{Der}\,A$ is a subspace. (easy)

2. Let D_1, D_2 be derivations. Need: $[D_1, D_2]$ is also.

$$[D_1, D_2](ab) \stackrel{?}{=} a \cdot [D_1, D_2](b) + [D_1, D_2](a) \cdot b \quad ?$$

LHS $(D_1 \circ D_2 - D_2 \circ D_1)(ab) = D_1 \circ D_2(ab) - D_2 \circ D_1(ab)$

$$= D_1(aD_2(b) + D_2(a)b) - D_2(aD_1(b) + D_1(a)b)$$

$$= a\, D_1 \circ D_2(b) + D_1(a) \circ D_2(b) + D_2(a) D_1(b) + D_1 D_2(a)\, b$$

etc.

Two Key Examples

1. $A = C^\infty(\mathbb{R})$ = smly diff functions $\mathbb{R} \to \mathbb{R}$

   Define $(fg)(x) = f(x)g(x)$, A is an assoc $\mathbb{R}$-alg

   Then $D(f) := f'$ is a derivation

2. $ad : L \to gl(L)$

   Image $\in Der(L) \subseteq gl(L)$

Problems

1.13, 1.15, 1.17

HW 1.9, 1.14, 1.18, 2.6, 2.8, 2.9, 2.11