Lecture 22

Review Let $R \subset E$ be a root system. A <u>base</u> of R is a basis B of E so that every $\alpha \in R$ is a $\mathbb{Z}$-linear combination of elements in B with all coefs ≥ 0 or all ≤ 0.

- Choice of B partitions roots R into positive roots R^+, neg roots R^-.
- Elements of B are called <u>simple roots</u>, $\{s_\alpha \mid \alpha \in B\}$ simple reflections.
- Every root system has a base, but there are many.

Ex $L = \mathfrak{sl}(3,\mathbb{C})$, $B = \{\alpha_1 = \varepsilon_1 - \varepsilon_2, \alpha_2 = \varepsilon_2 - \varepsilon_3\}$

$\Phi^+ = \{\alpha_1, \alpha_2, \alpha_1 + \alpha_2 = \varepsilon_1 - \varepsilon_3\}$, $\Phi^- = \{-\alpha_1, -\alpha_2, -\alpha_1 - \alpha_2\}$

Weyl Group $GL(E) = \{$invertible linear maps $E \to E\}$ general linear group.

Def The <u>Weyl group</u> $W = W(R)$ is the subgroup of $GL(E)$ generated by the reflections $\{s_\alpha \mid \alpha \in R\}$

Thm $W(R)$ is finite.

Pf The s_α permute roots, so we get a homomorphism

$$W(R) \to Sym(R) \leftarrow \text{order } |R|!$$

But any linear map fixing each $\alpha \in R$ is Id, since R spans E.
So $W(R) \cong$ to a subgroup of $Sym(R)$. //

Goal Given a base B, we can recover everything about R.

Rmk We saw this illustrated in the rank 2 examples last time.

<u>Lemma</u> Let $\alpha \in B$. Then $s_\alpha(\alpha) = -\alpha$, but s_α permutes the remaining positive roots amongst themselves.

<u>Pf</u> Choose $\beta \in R^+$, $\beta \neq \alpha$. Let $\beta = c_\alpha \alpha + c_\gamma \gamma + \dots$ some $\gamma \in B$ $c_\gamma > 0$. Then $s_\alpha(\beta) = \beta - \langle \beta, \alpha \rangle \alpha$ has same coeft of γ so is also positive. //

<u>Def</u> 1. Let $\beta = \sum_{\gamma \in B} c_\gamma \gamma$, be a positive root. Its <u>height</u>, $ht(\beta) = \sum_\gamma c_\gamma$.

2. Let $W_0 = \langle s_\gamma \mid \gamma \in B \rangle \leq W$, subgroup gen by simple reflections.

<u>Prop</u> Let $\beta \in R$. Then $\exists\, g \in W_0$ and $\alpha \in B$ with $\beta = g(\alpha)$.

<u>Pf</u> If $\beta = g(\alpha)$ then $-\beta = -g(\alpha) = g(-\alpha) = g \circ s_\alpha(\alpha)$ so WLOG $\beta \in R^+$.

Proceed by induction on $ht\,\beta$. $ht = 1$ means $\beta \in B$ so $\beta = id(\beta)$ ✓

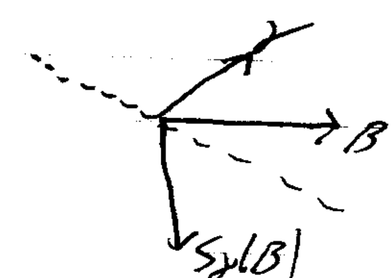
Assume $ht(\beta) \geq 2$. Then $\beta = \sum_{\gamma \in B} c_\gamma \gamma$. Note

$$0 < (\beta,\beta) = \sum_{\gamma \in B} c_\gamma (\beta,\gamma) \quad \text{so } \exists\, \gamma \in B \text{ with } (\beta,\gamma) > 0,\ \langle \beta,\gamma \rangle > 0.$$

Then $s_\gamma(\beta) = \beta - \langle \beta, \gamma \rangle \gamma$ has smaller height. Thus

$\exists\, \tilde{g} \in W_0$, $\alpha \in B$ with $\tilde{g}(\alpha) = s_\gamma(\beta)$ so $s_\gamma \circ \tilde{g}(\alpha) = \beta$ //

<u>Picture</u>: Find simple root γ w acute $\measuredangle$ with β, subtract off a multiple of it by applying s_γ



//

HW $\alpha \in R,\ g \in W \Rightarrow g s_\alpha g^{-1} = s_{g\alpha}$

COR $W_0 = W$, i.e. W is generated by simple reflections.

Pf Let $\beta \in R$ ETS $s_\beta \in W_0$. But $\beta = g(\alpha)$, some $\alpha \in B$, $g \in W_0$.
Then $s_\beta = s_{g(\alpha)} = g s_\alpha g^{-1} \in W_0$. //

Cartan Matrices and Dynkin Diagrams

Thm Let B, B' be two bases of R. Then $\exists g \in W(R)$ so $B' = \{g(\alpha) \mid \alpha \in B\}$
Pf APP D

Def Let $B = \{\alpha_1, \alpha_2, \ldots, \alpha_\ell\}$ be a base of R. The Cartan matrix of R is the integral $\ell \times \ell$ matrix with i,j entry $\langle \alpha_i, \alpha_j \rangle$.

Ex $sl(4, \mathbb{C})$, $B = \{\varepsilon_1 - \varepsilon_2,\ \varepsilon_2 - \varepsilon_3,\ \varepsilon_3 - \varepsilon_4\}$

$\langle \varepsilon_1 - \varepsilon_2, \varepsilon_1 - \varepsilon_2 \rangle = \frac{2(\varepsilon_1 - \varepsilon_2, \varepsilon_1 - \varepsilon_2)}{(\varepsilon_1 - \varepsilon_2, \varepsilon_1 - \varepsilon_2)} = 2$ * Diagonal entries always 2

$\langle \varepsilon_1 - \varepsilon_2, \varepsilon_2 - \varepsilon_3 \rangle = \frac{2(\varepsilon_1 - \varepsilon_2, \varepsilon_2 - \varepsilon_3)}{(\varepsilon_2 - \varepsilon_3, \varepsilon_2 - \varepsilon_3)} = \frac{2 \cdot -1}{2} = -1$ $\quad C = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}$

Ex B_2, base $\{\alpha, \beta\}$ $(\alpha,\alpha) = 1$, $(\beta,\beta) = 2$, $(\alpha,\beta) = -1$

$C = \begin{pmatrix} 2 & -1 \\ -2 & 2 \end{pmatrix}$

Rmk Reflections preserve lengths and angles. Thus

$$\langle s_\beta(\alpha_i), s_\beta(\alpha_j) \rangle = \langle \alpha_i, \alpha_j \rangle \quad \forall \beta \in R.$$

* C depends only on ordering of roots in base, not on actual choice of base. *

Dynkin Diagrams

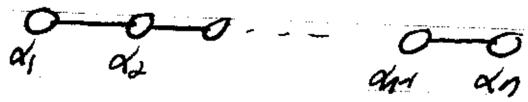
Def Given root system R and its Cartan Matrix, define a graph $\Delta = \Delta(R)$:

Vertices: simple roots

<u>Edges</u>: If $\langle \alpha, \beta \rangle \langle \beta, \alpha \rangle = d_{\alpha\beta} \neq 0$ (i.e. $\alpha \not\perp \beta$), draw $d_{\alpha\beta}$ edges. If $(\alpha,\alpha) \neq (\beta,\beta)$, point edge long $\to$ short.

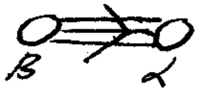
Δ is called the <u>Dynkin Diagram</u>

Ex $\mathfrak{sl}(n+1, \mathbb{C})$, simple roots $\alpha_1 = \varepsilon_1 - \varepsilon_2, \alpha_2 = \varepsilon_2 - \varepsilon_3, \ldots, \alpha_n = \varepsilon_n - \varepsilon_{n-1}$



A_n

Ex B_2 base α, β, recall $\langle \alpha, \beta \rangle \langle \beta, \alpha \rangle = 3$



G_2

Ex $\mathfrak{so}(4, \mathbb{C}) \cong \mathfrak{sl}(2, \mathbb{C}) \oplus \mathfrak{sl}(2, \mathbb{C})$

$A_1 \times A_2$

<u>Easy Exercise</u> R is irreducible $\longleftrightarrow$ $\Delta(R)$ is connected

<u>Next Goal</u> $\Delta(R)$ determines R up to $\cong$.

Def Let $R \subset E$, $R' \subset E'$ be root systems. Say they are <u>isomorphic</u> if $\exists$ a linear isomorphism $\varphi : E \to E'$ s.t.

1. $\varphi(R) = R'$

2. $\langle \alpha, \beta \rangle = \langle \varphi(\alpha), \varphi(\beta) \rangle \ \forall \alpha, \beta \in R$

* φ preserves $\measuredangle$s and relative lengths.

Ex 1 $s_\alpha : \alpha \in R$ is an $\cong$ from R to itself.

Ex 2 Given $R \subset E$, fix $c \in \mathbb{R}$. Then $\{c\alpha \in E\}$ also a root system and $\vec{v} \to c\vec{v}$ is an $\cong$.

Thm Two root systems $R \subset E$ and $R' \subset E'$ are $\cong$ iff they have the same Dynkin Diagram.

Pf $\Longrightarrow$ Clear, Cartan matrix is preserved.

$\Longleftarrow$

Choose bases $B = \{\alpha_1, \alpha_2, \ldots, \alpha_\ell\}$ in R, $B' = \{\alpha_1', \alpha_2', \ldots, \alpha_\ell'\}$ in R' so Dynkin Diagrams line up, i.e.

$$(*) \quad \langle \alpha_i, \alpha_j \rangle = \langle \alpha_i', \alpha_j' \rangle \quad \forall i, j.$$

Let $\varphi : E \to E'$ be the unique linear map with $\varphi(\alpha_i) = \alpha_i'$ $\forall i$.

Remains to show $\varphi(R) = R'$.

Let $v \in E$, $\alpha_i \in B$. $\quad \varphi(s_{\alpha_i}(v)) = \varphi(v - \langle v, \alpha_i \rangle \alpha_i)$

$$= \varphi(v) - \langle v, \alpha_i \rangle \alpha_i'$$

$$= \varphi(v) - \langle \varphi(v), \alpha_i' \rangle \alpha_i' \quad \text{using } (*)$$

$$= s_{\alpha_i'}(\varphi(v))$$

Thus $\quad \varphi(s_{\alpha_i}(v)) = s_{\alpha_i'}(\varphi(v))$

applying simple reflections then φ = applying φ then simple reflections

$\rightsquigarrow \varphi(R) = R'$

since R is image of B under simple reflections, similarly R', B' //