Lecture 20

Review L semisimple Lie alg $/\mathbb{C}$, H a Cartan subalgebra. Root space decomp:

$$L = H \oplus \bigoplus_{\alpha \in \Phi} L_\alpha \quad \text{as } H\text{-modules}, \quad \Phi \subseteq H^* \text{ set of roots}$$

Killing form κ is nondeg on H, so induces $H \to H^*$ by $h \to \kappa(h,-)$.

Def For $\alpha \in H^*$ let t_α be corresponding vector in H, so $\alpha(h) = \kappa(t_\alpha, h)$.

Summary of Results so Far

1. Φ spans H^*. Thus for $0 \neq h \in H$, $\exists\, \alpha \in \Phi$ w/ $\alpha(h) \neq 0$.

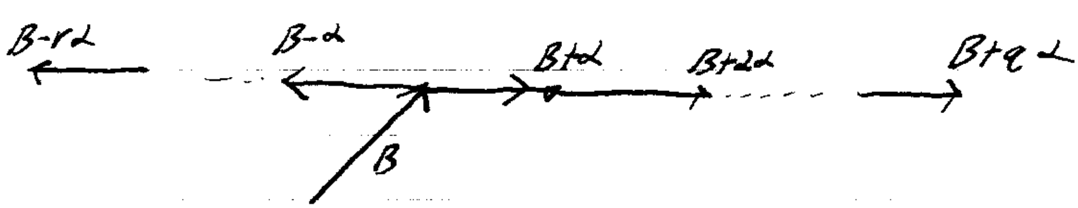
2. Let $\alpha \in \Phi$. Then $\dim L_\alpha = 1$ and $\pm\alpha$ are only multiples of α in Φ.

3. $L_\alpha \perp L_\beta$ $(\kappa(L_\alpha, L_\beta) = 0)$ unless $\beta = -\alpha$ in which case $\kappa(L_\alpha, L_{-\alpha}) \neq 0$.

4. $x \in L_\alpha$, $y \in L_{-\alpha} \Rightarrow [x,y] = \kappa(x,y)\, t_\alpha$, $[L_\alpha, L_{-\alpha}] = \operatorname{span}\{t_\alpha\}$

5. For $\alpha \in \Phi$ $\exists$ $e_\alpha \in L_\alpha$, $f_\alpha \in L_{-\alpha}$, $h_\alpha = [e_\alpha, f_\alpha]$ w/ $\alpha(h_\alpha) = 2$ and $sl(\alpha) := \langle e_\alpha, f_\alpha, h_\alpha \rangle \cong sl(2, \mathbb{C})$ by obvious map.

6. Let $\alpha, \beta \in \Phi$. Consider α root string through β:



Then $\exists\, r, q \in \mathbb{Z}^{\geq 0}$ so $\beta + i\alpha \in \Phi$ iff $-r \leq i \leq q$.

Moreover $r - q = \beta(h_\alpha)$.

7. $M = \bigoplus_{i \in \mathbb{Z}} L_{\beta + i\alpha}$, for $\beta \neq \pm\alpha$. Then M is an irreducible $sl(\alpha)$-module

<u>Prop</u> $\alpha, \beta \in \Phi$, $\beta \neq \pm\alpha$

8. $\beta(h_\alpha) \in \mathbb{Z}$
9. If $\alpha+\beta \in \Phi$ then $[e_\alpha e_\beta] \neq 0$ (hence is a mult of $e_{\alpha+\beta}$)
10. $\beta - \beta(h_\alpha)\alpha \in \Phi$.

<u>Pf</u> 8. Consider $x \in L_\beta \subseteq M$. Then $[h_\alpha, x] = \beta(h_\alpha)x$ so $\beta(h_\alpha) \in \mathbb{Z}$.

9. Suppose $[e_\alpha, e_\beta] = 0$. But $[h_\alpha, e_\beta] = \beta(h_\alpha) e_\beta$ so e_β is a highest weight vector in M.

But for $y \in L_{\alpha+\beta}$, $[h_\alpha, y] = (\alpha+\beta)(h_\alpha) y = (\alpha(h_\alpha) + \beta(h_\alpha))y$

$= (2 + \beta(h_\alpha))y$ $\divideontimes$,

10. Use #6, $\beta(h_\alpha) = r - q$ so $-r \leq -(r-q) \leq q$ //

<u>Techical Lemma</u> Let $\alpha \in \Phi$

1. $t_\alpha = \frac{h_\alpha}{\kappa(e_\alpha, f_\alpha)}$, $h_\alpha = \frac{2t_\alpha}{\kappa(t_\alpha, t_\alpha)}$

2. $\kappa(t_\alpha, t_\alpha)\kappa(h_\alpha, h_\alpha) = 4$

<u>Pf</u>

1. $h_\alpha = [e_\alpha, f_\alpha] = \kappa(e_\alpha, f_\alpha) t_\alpha$ by #4.

1. $2 = \alpha(h_\alpha) = \kappa(t_\alpha, h_\alpha) = \kappa(t_\alpha, \kappa(e_\alpha, f_\alpha) t_\alpha)$ so

$2 = \kappa(t_\alpha, t_\alpha)\kappa(e_\alpha, f_\alpha)$

$2 = \kappa(t_\alpha, t_\alpha) \frac{h_\alpha}{t_\alpha}$ //

2. $\kappa(h_\alpha, h_\alpha) = \kappa\left(\frac{2t_\alpha}{\kappa(t_\alpha, t_\alpha)}, \frac{2t_\alpha}{\kappa(t_\alpha, t_\alpha)}\right)$

$= \frac{4}{\kappa(t_\alpha, t_\alpha)}$ //

Cor Let $\alpha, \beta \in \Phi$. Then $\kappa(h_\alpha, h_\beta) \in \mathbb{Z}$, $\kappa(t_\alpha, t_\beta) \in \mathbb{Q}$

Pf Use $L = H \oplus \bigoplus_{\gamma \in \Phi} L_\gamma$, $\mathrm{ad}\, h_\alpha$, $\mathrm{ad}\, h_\beta$ are diagonal: $\mathrm{ad}\, h_\alpha = \begin{pmatrix} 0 & & & \\ & 0 & & \\ & & 0 & \\ & & & \gamma(h_\alpha) \ddots \end{pmatrix}$

1. $\kappa(h_\alpha, h_\beta) = \sum_{\gamma \in \Phi} \gamma(h_\alpha)\gamma(h_\beta) \in \mathbb{Z}$ since $\gamma(h_{\alpha,\beta})$ are eigenvalues of $h_{\alpha,\beta}$ so in $\mathbb{Z}$.

2. Exercise, use Tech Lemma, $\in \langle \frac{4}{\kappa(h_\alpha, h_\beta)}\text{'s} \rangle$ //

Key Def Define a nondegenerate, symmetric, bilinear form, $(\ ,\ )$ on H^* by

$$(\theta, \varphi) = \kappa(t_\theta, t_\varphi) \quad \forall \theta, \varphi \in H^*.$$

Prop

1. $(\alpha, \beta) \in \mathbb{Q} \quad \forall \alpha, \beta \in \Phi$
2. $\beta(h_\alpha) = \dfrac{2(\beta, \alpha)}{(\alpha, \alpha)}$
3. Choose a basis $\{\alpha_1, \alpha_2, \ldots, \alpha_\ell\}$ of H^* with $\alpha_i \in \Phi$.
   Then for any $\beta \in \Phi$, β is a $\mathbb{Q}$-linear comb. of the $\{\alpha_i\}$.

Pf

1. $(\alpha, \beta) = \kappa(t_\alpha, t_\beta) \in \mathbb{Q}$

2. $\beta(h_\alpha) = \kappa(t_\beta, h_\alpha) = \kappa\left(t_\beta, \dfrac{2 t_\alpha}{(t_\alpha, t_\alpha)}\right) = \dfrac{2(\beta, \alpha)}{(\alpha, \alpha)}$

3. Let $\beta = \sum_{i=1}^{\ell} c_i \alpha_i$, $c_i \in \mathbb{C}$. Then

$$(\beta, \alpha_j) = \sum_{i=1}^{\ell} (\alpha_i, \alpha_j) c_i$$

invertible w/ rational entries

all in $\mathbb{Q}$ by Prop 1

$$\begin{pmatrix} (\beta, \alpha_1) \\ (\beta, \alpha_2) \\ \vdots \\ (\beta, \alpha_\ell) \end{pmatrix} = \begin{pmatrix} (\alpha_1, \alpha_1) & \cdots & (\alpha_\ell, \alpha_1) \\ \vdots & & \\ (\alpha_1, \alpha_\ell) & \cdots & (\alpha_\ell, \alpha_\ell) \end{pmatrix} \begin{pmatrix} c_1 \\ \vdots \\ c_\ell \end{pmatrix}$$

Def Let $E \subseteq H^*$ be the $\mathbb{R}$ span of $\{\alpha_1, \alpha_2, \ldots, \alpha_\ell\}$, so $\Phi \subseteq E$.

Prop $(\,,\,)$ is an $\mathbb{R}$-valued inner product on E.

Pf ETS positive definite. Let $\theta \in E \subseteq H^*$. Then:

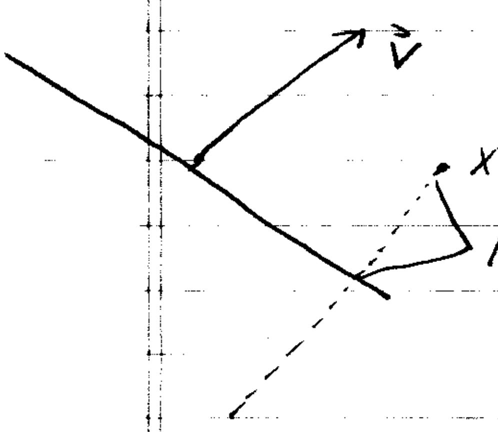
$$(\theta,\theta) = \kappa(t_\theta, t_\theta) = \sum_{\gamma\in\Phi} \gamma(t_\theta)^2 = \sum_{\gamma\in\Phi} \kappa(t_\gamma, t_\theta)^2 = \sum_{\gamma\in\Phi} (\gamma,\theta)^2 \geq 0$$

This $=0$ iff $\gamma(t_\theta)=0\ \forall \gamma \Leftrightarrow t_\theta = 0 \Leftrightarrow \theta = 0$. //

Review (Reflections) Suppose E is a f.d. vector space w/ $(\,,\,)$ inner product. Let $0 \neq v \in E$. Let s_v be reflection in hyperplane $\perp$ to v. Then

$$s_v(x) = x - \frac{2(x,v)}{(v,v)} v$$

Notation $\langle x, v\rangle = \frac{2(x,v)}{(v,v)}$ so $s_v(x) = x - \langle x,v\rangle v$



length is $x \cdot \frac{v}{|v|}$ so vector is $\frac{x\cdot v}{|v|}\frac{v}{|v|} = \frac{(x,v)}{(v,v)} v$

Rmk

1. $s_v(v) = -v$, $s_v(y) = y\ \forall y \perp v$

2. $(s_v^2) = s_v \circ s_v = Id$

<u>Def</u> Let E be a real vector space w/ $(\,,\,)$ inner product. $R \subset E$ is a <u>root system</u> if

(R1) R is finite, $0 \notin R$
(R2) If $\alpha \in R$ then only scalar multiples of $\alpha \in R$ are $\pm\alpha$
(R3) For $\alpha \in R$, s_α permutes elements of R (i.e. $s_\alpha(R) = R$
(R4) $\alpha, \beta \in R$ then $\langle \beta, \alpha \rangle \in \mathbb{Z}$.

<u>Key Example</u> E is $\mathbb{R}$-span of Φ in H^*

R1 ✓
R2 ✓
<u>R3</u> $s_\alpha(\beta) = \beta - \beta(h_\alpha)\alpha \in \Phi$ by 10.

<u>R4</u> $\langle \beta, \alpha \rangle = \beta(h_\alpha) \in \mathbb{Z}$ by 8.

<u>Next ~~two~~ 4++ classes</u> 1. Classify all root systems by Dynkin Diagrams

2. Find complex semisimple L realizing each.

<u>Rmk</u> 1. (R2) sometimes called reduced root system

2. Note only <u>ratios</u> of $(\,,\,)$ occur.