Lecture 19 10.5, 10.7, 10.8, 10.9

Review L a s.s Lie alg $/\mathbb{C}$, last time we proved L contains a Cartan subalgebra H with $H = C_L(H)$. Thus we obtained:

$$(*) \quad L = H \oplus \bigoplus_{\alpha \in \Phi} L_\alpha \qquad \text{root space decomposition}$$

where $\Phi \subseteq H^*$ is the set of roots and $L_\alpha = \{x \mid [h,x] = \alpha(h)x \ \forall h \in H\}$

Prop 1. $[L_\alpha, L_\beta] \subseteq L_{\alpha+\beta} \quad \forall \alpha,\beta \in H^*$
2. If $\alpha+\beta \neq 0$ then $L_\alpha \perp L_\beta$ (i.e. $\kappa(L_\alpha, L_\beta) = 0$)
3. κ is nondegenerate on H.

HW $(,)$ nondeg bil. form on V induces $\cong V \to V^*$ given by $v \to (v, -)$

Ex Killing form on H, for $\alpha \in H^*$ let t_α be corr. elt of H so

$$(**) \qquad \alpha(h) = \kappa(t_\alpha, h) \ \forall h \in H$$

Weyl's Thm L s.s. $/\mathbb{C}$. Every f.d. L-module is comp reducible.
pf See appendix.

Thm Let $\alpha \in \Phi$. Then $\exists\, x \in L_\alpha, y \in L_{-\alpha}$ so $\{x, y, [x,y]\}$ is $\cong$ to $\mathfrak{sl}(2, \mathbb{C})$.

Pf Choose $0 \neq x \in L_\alpha$. By Prop(2) and (3), $\exists\, 0 \neq y \in L_{-\alpha}$ with $\kappa(x,y) \neq 0$.

Claim $[x,y] \neq 0$.
Choose $t \in H$ w/ $\alpha(t) \neq 0$. Then $[t,x] = \alpha(t)x$ so

$$0 \neq \alpha(t)\kappa(x,y) = \kappa([t,x],y) = \kappa(t,[x,y])$$
so $0 \neq [x,y] \in H$.

Since $[x,y]\in H$, $[[x,y],x]=\alpha([x,y])x$, $[[x,y],y]=-\alpha([x,y])y$, so $S=\langle x,y,[x,y]\rangle$ is a subalg. Let $h=[x,y]$.

If $\alpha(h)\neq 0$ then h commutes w/ x,y $\overset{5.7}{\Longrightarrow}$ h is ad-nilp. But h is semisimple ⨳. Thus $\alpha(h)\neq 0$ so $\{x,y,x\}\in S' \Rightarrow S\cong sl(2,\mathbb{C})$ //

Rmk Adjusting y by a scalar if nec, we can get

$$sl(2,\mathbb{C})\cong\langle e_\alpha, f_\alpha, h_\alpha=[e_\alpha,f_\alpha]\rangle \text{ with } \alpha(h_\alpha)=2$$

just as in std $sl(2,\mathbb{C})$ w/ $\alpha=\varepsilon_1-\varepsilon_2$, call it $sl(\alpha)$.

Ex $L=sl(3,\mathbb{C})$ $\alpha=\varepsilon_1-\varepsilon_3$ $e_\alpha=\begin{pmatrix}0&0&1\\0&0&0\\0&0&0\end{pmatrix}$ $f_\alpha=\begin{pmatrix}0&0&0\\0&0&0\\1&0&0\end{pmatrix}$ $h_\alpha=\begin{pmatrix}1&0&0\\0&0&0\\0&0&-1\end{pmatrix}$

EX Let $\beta\in\Phi$ or $\beta=0$. Let $M=\bigoplus_{\substack{c\in\mathbb{C}\\ \text{w/ } \beta+c\alpha\in\Phi}} L_{\beta+c\alpha}$.

Then M is an $sl(\alpha)$-submodule of L, called α-root string through β.

Pf $[L_{\pm\alpha}, L_{\beta+c\alpha}]\subseteq L_{\beta+(c\pm1)\alpha}$.

Thm (Basic Properties of Root Space Decomp)

1. Φ spans H^*
2. $\alpha\in\Phi \rightarrow -\alpha\in\Phi$
3. $\alpha\in\Phi$, $x\in L_\alpha$, $y\in L_{-\alpha}$. Then $[x,y]=\kappa(x,y)\,t_\alpha$
4. $\alpha\in\Phi$ then $[L_\alpha,L_{-\alpha}]=\text{span}\{t_\alpha\}$
5. $\alpha(t_\alpha)=\kappa(t_\alpha,t_\alpha)\neq 0$ $\forall\alpha\in\Phi$

Proof

1. Suppose not, $\exists\ 0 \neq h \in H$ s.t. $\alpha(h) = 0\ \forall \alpha \in \Phi$. Thus $[h, L_\alpha] = 0\ \forall \alpha \in \Phi$
$\Rightarrow h \in Z(L)$ $\divideontimes$.

2. As before, follows from nondegeneracy since $L_\alpha \perp$ all other L_β except $\beta = -\alpha$.

3. $$\begin{aligned}\kappa(h, [x,y]) = \kappa([h,x], y) &= \alpha(h)\kappa(x,y) \\ &= \kappa(t_\alpha, h)\kappa(x,y) \\ &= \kappa(\kappa(x,y)t_\alpha, h) = \kappa(h, \kappa(x,y)t_\alpha)\end{aligned}$$
Thus $[x,y] - \kappa(x,y)t_\alpha \in H \cap H^\perp = 0$. //

4. Just need some $x \in L_\alpha, y \in L_{-\alpha}$ w/ $\kappa(x,y) \neq 0$, as before from nondeg.

5. $\alpha(t_\alpha) = \kappa(t_\alpha, t_\alpha)$ by def. Suppose it is 0. Then $[t_\alpha, x] = [t_\alpha, y] = 0$
$\forall x \in L_\alpha, y \in L_{-\alpha}$.
As before choose $x \in L_\alpha, y \in L_{-\alpha}$, $\kappa(x,y) = 1$. Then

by (3), $[x,y] = t_\alpha$ so $\langle x, y, [x,y] \rangle$ is solvable subalg,
so $t_\alpha = [x,y]$ is ad-nilpotent. But t_α is semisimple!
Thus $\operatorname{ad} t_\alpha = 0 \Rightarrow t_\alpha \in Z(L)$ $\divideontimes$. //

Integrality Properties

Thm

1. $\alpha \in \Phi \Rightarrow \dim L_\alpha = 1$

2. $\alpha \in \Phi \Rightarrow$ only scalar multiples of α that are roots are $\pm \alpha$.

3. $\exists$ integers $r, q \geq 0$ such that for $k \in \mathbb{Z}$, $\beta + k\alpha \in \Phi$ iff $-r \leq k \leq q$.
Moreover $r - q = \beta(h_\alpha)$

Proof Fix $\alpha \in \Phi$, let $sl(\alpha) = \langle e_\alpha, f_\alpha, h_\alpha \rangle$

Define $K = \ker \alpha \in H$. Since $\alpha : H \to \mathbb{C}$, $\dim K = \dim H - 1$.

Claim K is an $sl(\alpha)$ submodule

Pf. $[e_\alpha, K] = [K, e_\alpha] = \alpha(K) e_\alpha = 0$, same for f_α. Thus //

Let $$M = H \oplus \bigoplus_{c \in \mathbb{C}^*} L_{c\alpha},$$ α root string through 0. Then

$$M \cong sl(\alpha) \oplus K \oplus \bigoplus_{c \in \mathbb{C}^* - \{\pm 1\}} L_{c\alpha}$$ as $sl(\alpha)$ modules.

* All 0 weights in 1st two terms, but any V_{2d} has a zero weight. Thus

* Only even weights for h_α on M are $\{0, \pm 2\}$

Cor 2α is not a root.

Pf. $x \in L_{2\alpha} \Rightarrow h_\alpha(x) = 2\alpha(h_\alpha) x = 4x$.

Cor $\frac{1}{2}\alpha$ is not a root, thus "1" is not a weight of h_α on M.

<u>Conclude</u> $M = H + sl(\alpha)$, so $\dim L_\alpha = 1$.

Thus 1 & 2 hold.

<u>Pf of 3</u> Let $\beta \neq \pm\alpha$.

Let $U = \sum_{i \in \mathbb{Z}} L_{\beta + i\alpha}$, so no $\beta + i\alpha = 0$.

Let $x \in L_{\beta + i\alpha}$. Then $h_\alpha(x) = (\beta + i\alpha)(h_\alpha)\, x = (\beta(h_\alpha) + 2i)\, x$.

Thus U has distinct 1-dim weight spaces $\beta(h_\alpha) + 2i$ for h_α.

Can't have 0 & 1 so U is irreducible, $U = V_d$