Lecture 18

Review. L complex, semisimple. Any $x \in L$ has an abstract Jordan dec $x = d + n$ with $[d, n] = 0$, ad d diagonalizable, ad n nilpotent.

1. $\text{ad}\,x = \text{ad}\,d + \text{ad}\,n$ is usual J.D. in $gl(L)$
2. If ad x is diagonalizable, $(n=0)$, say x is a semisimple element.

Goal: Classify complex semisimple Lie algebras

Idea
1. Suppose $H \leq L$ is abelian w/ all elts semisimple, so all ad h are diagonaliz.
2. $[\text{ad}\,h, \text{ad}\,\tilde{h}] = \text{ad}\,[h, \tilde{h}] = 0$, so all ad h are simultaneously diag.
3. Thus L has a basis of weight vectors for adjoint action of H.
4. When H is "large" this lets us classify L.

Motivating Ex $L = sl(3, \mathbb{C})$, H = 2-dim subalg of diagonal matrices.

Exercise Let $h = \begin{pmatrix} a_1 & 0 & 0 \\ 0 & a_2 & 0 \\ 0 & 0 & a_3 \end{pmatrix}$, then $[h, e_{ij}] = (a_i - a_j) e_{ij}$.

Def $\varepsilon_i \in H^*$ by $\varepsilon_i(h) = a_i$

Rmk 1. Let L_{ij}, $i \neq j$, be span$\{e_{ij}\}$. Then L_{ij} is a weight space for H with weight $\varepsilon_i - \varepsilon_j$.

2. H is a 2-dim'l weight space for zero weight

3. Thus $sl(3, \mathbb{C}) \cong H \oplus \bigoplus_{i \neq j} L_{ij}$ as H-modules

4. $[\,,\,]$ and wts are compatible, for example

e_{12} wt $\varepsilon_1 - \varepsilon_2$
e_{23} wt $\varepsilon_2 - \varepsilon_3$
$[e_{12}, e_{23}]$ wt $\varepsilon_1 - \varepsilon_3$

Prelims

Setup H abelian subalg, all elts s.s. So L has a basis of weight vectors under adjoint action of H on L.

Def For $\alpha \in H^*$ let $L_\alpha = \{x \in L \mid [h,x] = \alpha(h)x \ \forall h \in H\}$ be the corresponding wt space

Ex $C_L(H) = \{x \in L \mid [h,x] = 0 \ \forall h \in H\}$ centralizer of H in L (check ideal if H is)
$= L_0$, note $H \leq L_0$ since abelian

Def Let $\Phi = \{0 \neq \alpha \in H^* \mid L_\alpha \neq 0\}$. Then:

$$L = L_0 \oplus \bigoplus_{\alpha \in \Phi} L_\alpha \qquad (*)$$

- Finite set Φ since L is f.d
- $L = \mathfrak{sl}(3,\mathbb{C})$ as above, $\Phi = \{\varepsilon_1 - \varepsilon_2, \varepsilon_1 - \varepsilon_3, \varepsilon_2 - \varepsilon_3, \varepsilon_2 - \varepsilon_1, \varepsilon_3 - \varepsilon_1, \varepsilon_3 - \varepsilon_2\}$

Lemma Let $\alpha, \beta \in H^*$

1. $[L_\alpha, L_\beta] \subseteq L_{\alpha+\beta}$
2. If $\alpha + \beta \neq 0$ then $\kappa(L_\alpha, L_\beta) = 0$ (i.e. $L_\alpha \perp L_\beta$)
3. κ is nondeg on L_0, i.e. $L_0 \cap L_0^\perp = 0$

Pf 1. Let $x \in L_\alpha, y \in L_\beta$

$$[h,[x,y]] = [[h,x],y] + [x,[h,y]] = [\alpha(h)x, y] + [x, \beta(h)y] = (\alpha(h)+\beta(h))[x,y] \quad //$$

2. $\alpha + \beta \neq 0 \Rightarrow \exists h \neq 0$ so $(\alpha+\beta)(h) \neq 0$. Let $x \in L_\alpha, y \in L_\beta$

$$\underbrace{\kappa([h,x],y)}_{\alpha(h)\kappa(x,y)} = -\kappa([x,h],y) = -\kappa(x,\underbrace{[h,y]}_{}) \quad -\beta(h)\kappa(x,y)$$

$$0 = (\alpha(h)+\beta(h))\kappa(x,y) \Rightarrow \kappa(x,y) = 0.$$

3. Suppose $z \in L_0 \cap L_0^\perp$. By (*) and 2, $z \in L^\perp \overset{ss}{\Rightarrow} z = 0$. //

Rmk We want H as large as possible, this makes Φ larger and wt spaces smaller, so Lemma gives more info on structure of L.

Def A Lie subalgebra H of L is a Cartan subalgebra (CSA) if (L is arbitrary)

1. H is abelian.
2. Every $h \in H$ is semisimple.
3. H is maximal wrt properties 1, 2 ($\not\exists H \subset \tilde{H}$, w/ $\tilde{H}$ as in 1, 2)

Rmk CSA aka maximal torus, maximal toral subalg.

Ex Diagonal matrices $h \subseteq sl(n, \mathbb{C})$ of dim 1.

$*$ $C_{sl(n,\mathbb{C})}(h) = h$ so h does not even sit in a larger abelian subalg.

Goal Let L be a complex semisimple Lie alg. Then L has a nonzero CSA H such that $H = C_L(H)$

Step 1 L contains a nonzero semisimple elt.

Pf Every x has abst J.D. $x = d + n$ with ad n nilpotent, $d, n \in L$. If all d's are 0 then every x is ad-nilpotent $\Rightarrow L$ is nilp $\Rightarrow L$ solv $\times$

Step 2 L contains a CSA

Pf Take $0 \neq s$ a semisimple elt of L. Choose $s \in H$ with H maximal subject to being abelian and consisting of semisimple elts. Then H is a CSA.

<u>Rmk</u> $C_L(H) = \bigcap_{h \in H} C_L(h)$

<u>Technical Lemma</u> Let H be CSA of ss Lie alg L. Then $\exists\, h \in H$ w $C_L(H) = C_L(h)$

Step 3 <u>Thm</u> Let H be a CSA of semisimple Lie alg L. Then $H = C_L(H)$!

<u>Proof</u> H is abelian so $H \subseteq C_L(H)$. Choose h so $C_L(H) = C_L(h)$.

<u>Claim</u> $C_L(h) \subseteq H$.

<u>Pf</u> Let $x = s + n$, $x \in C_L(h)$!

Note $[h,x] = 0 \Rightarrow [h,s] = [h,n] = 0$ so $s, n \in C_L(h)$.
Thus $s \in H$, otherwise $\langle H, s \rangle$ is larger CSA.

Since $s \in H$, $\operatorname{ad} s|_{C_L(h)} = 0$, thus $\operatorname{ad} x = \operatorname{ad} n : C_L(h) \to C_L(h)$.

So $\forall x \in C_L(h)$, $\operatorname{ad} x \in \mathfrak{gl}(C_L(h))$ is nilpotent $\overset{\text{Engel}}{\Longrightarrow}$ <u>$C_L(h)$ is nilpotent.</u>

Consider $\operatorname{ad} : C_L(h) \to \mathfrak{gl}(L)$, image is solvable, so $\exists$ basis of L so all upper Δ, and $\operatorname{ad} n$ strictly. Thus:

$$\forall y \in C_L(h), \quad \kappa(n,y) = \operatorname{tr}(\operatorname{ad} n \circ \operatorname{ad} y) = 0$$

$$\Rightarrow n \in L_0 \cap L^{\perp} \overset{\text{Lemma pt 3}}{\Longrightarrow} n = 0$$

Thus $x = s \in H$ so $C_L(h) \subseteq H$, i.e. $\boxed{H = C_L(H)}$

Summary

L semisimple Lie algebra $/\mathbb{C}$, $H = C_L(H)$ a CSA.

Since $L_0 = C_L(H) = L$ we have

$$L = H \oplus \bigoplus_{\alpha \in \Phi} L_\alpha \quad \leftarrow \text{root space decomposition}$$

Def

$\Phi \subset H^*$ are the set of roots of L

$\alpha \in \Phi$, L_α is the corresponding root space

Rmks 1. "root" = "weight for H"

2. Decomposition depends on choice of H.