Lecture 16

Bilinear Forms Def V a vector space, a map $(,): V\times V \to F$, linear in each coordinate, is a bilinear form.

Def Let V have a basis $B=\{v_1, v_2, \dots, v_n\}$. The Gram matrix of $(,)$ is $A=(a_{ij})$ where $a_{ij}=(v_i, v_j)$

Remark 1. Suppose $[v]_B = \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \vdots \\ \lambda_n \end{pmatrix}$, i.e. $v=\sum \lambda_i v_i$. Then

$$(v,w) = [v]_B^{tr} A [w].$$

2. $\{(v_i, v_j)\}$ defines uniquely a bilinear form as above.

Def Given any subset $U \subseteq V$, $U^\perp = \{v \in V \mid (u,v)=0 \ \forall u \in U\}$, is always a subspace.

Def $(,)$ is nondegenerate if $V^\perp = \{0\}$.

Exercise Let W be a subspace and $(,)$ be nondegenerate. Then

$$\dim W + \dim W^\perp = \dim V.$$

Pf $v \to (v, -)$ gives a linear map $V \to W^*$ w/ ker $W^\perp$.

Example $L/\mathbb{C}$ a Lie Algebra, the Killing Form $\kappa(x,y) = tr(ad\,x \circ ad\,y)$

- κ is symmetric
- $\kappa([x,y],z) = \kappa(x,[y,z]) \ \forall x,y,z \in L$ "associative"

<u>Cartan's 1st Criterion</u> $L/\mathbb{C}$ is solvable if & only iff $L' \subseteq L^{\perp}$.
(i.e. $\kappa(x,y)=0 \ \forall x \in L, y \in L'$)

<u>Example</u> $L = \mathfrak{sl}(2,\mathbb{C})$, basis $\{e,f,h\}$ $\operatorname{ad} e = \begin{pmatrix} 0 & 0 & -2 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$ $\operatorname{ad} f = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 2 \\ -1 & 0 & 0 \end{pmatrix}$ $\operatorname{ad} h = \begin{pmatrix} 2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

<u>Exercise</u> Gram matrix of κ: $\begin{pmatrix} 0 & 4 & 0 \\ 4 & 0 & 0 \\ 0 & 0 & 8 \end{pmatrix}$, not solvable!

<u>Ex</u> $\mathfrak{b} = \langle h, e \mid [h,e] = 2e \rangle$ $\operatorname{ad} e = \begin{pmatrix} 0 & -2 \\ 0 & 0 \end{pmatrix}$ $\operatorname{ad} h = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}$, $\kappa(h,h) = 4$

<u>Warning</u> $L = \mathfrak{sl}(2,\mathbb{C})$, $\kappa_L(h,h) \neq \kappa_{\mathfrak{b}}(h,h)$.

<u>Prop</u> Suppose $I \subset L$ is an ideal. Then $\forall x,y \in I$, $\kappa_L(x,y) = \kappa_I(x,y)$.

<u>Pf</u> Choose a basis of I and extend. Note $\operatorname{ad} x(L), \operatorname{ad} y(L) \subseteq I$.

On L: Matrix of $\operatorname{ad} x = \begin{pmatrix} A_x & B_x \\ 0 & 0 \end{pmatrix}$, of $\operatorname{ad} y$ $\begin{pmatrix} A_y & B_y \\ 0 & 0 \end{pmatrix}$
On I: " " " $= (A_x)$ " " $(\cancel{B}A_y)$

$$\kappa_L(x,y) = tr\begin{pmatrix} A_xA_y & A_xB_y \\ 0 & 0 \end{pmatrix} = tr(A_xA_y) = \kappa_I(x,y) \quad //$$

<u>Rmk</u> Solvable and semisimple are in some sense opposite extremes, so the following is not surprising.

Thm (Cartan's 2nd Criterion) $L/\mathbb{C}$ is semisimple if & only if κ_L is nondegenerate.

Proof $\Rightarrow$ Assume L is semisimple.

Lemma Suppose $I \subset L$ is an ideal. So is $I^\perp$ (wrt κ)

Pf Let $x \in L$, $y \in I^\perp$, $i \in I$. $\kappa([y,x], i) = \kappa(y, [x,i])$
$= \kappa(y, \tilde{i}) = 0$ so $[y,x] \in I^\perp$ //

Consider the ideal $L^\perp$, we must show it is zero. Now $\kappa(x,y) = 0$ $\forall x,y \in L^\perp$
so $L^\perp$ is solvable by 1st criterion $\Rightarrow L^\perp = 0$ by semisimplicity.

$\Leftarrow$ Suppose L is not semisimple, so by HW L has a nonzero abelian ideal A. We show $A \subseteq L^\perp$.

$\forall a \in A$, $x \in L$ $[x,a] \in A$ so $[a,[x,a]] = 0$. Thus

$$\operatorname{ad} a \circ \operatorname{ad} x \circ \operatorname{ad} a = 0 \Rightarrow (\operatorname{ad} a \circ \operatorname{ad} x)^2 = 0$$

$\Rightarrow \operatorname{ad} a \circ \operatorname{ad} x$ is nilpotent $\Rightarrow \kappa(a,x) = 0$.

Thus $A \subseteq L^\perp$ so κ is degenerate. //

Now an important application...

Lemma Suppose L is semisimple, $0 \subset I \subset L$. Then $L = I \oplus I^\perp$ and I is semisimple.

Proof As above, $I \cap I^\perp$ is solvable by 1st crit, so $I \cap I^\perp = 0$ so $L = I \oplus I^\perp$.

Now apply 2nd criterion to I. If no then κ_I is degenerate. Choose
$0 \neq a \in I$ with $\kappa(a,i) = 0$ $\forall i \in I$. But $\kappa(a,*) = 0$ $\forall * \in I^\perp$
so $a \in L^\perp$, κ is degenerate. $\cancel{=}$

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<u>Recall</u> Semisimple = no solvable ideals, simple = nonabelian w/ no nontrivial ideals

<u>Thm</u> L a Lie alg/$\mathbb{C}$. Then L is semisimple $\longleftrightarrow$ $\exists$ simple ideals $L_1, L_2, .., L_r$ so

$$L = L_1 \oplus L_2 \oplus \cdots \oplus L_r.$$

$\Longrightarrow$

<u>Proof</u> Let L be semisimple, proof by induction on $\dim L$. If L simple, done.

Otherwise choose $0 \subset I \subset L$, so I is simple (we know I is nonabelian since L ss). Thus $L = I \oplus I^\perp$ and $I^\perp$ semisimple by Lemma. Thus

$$L = I \oplus I^\perp, \quad I^\perp = L_2 \oplus L_3 \oplus \cdots \oplus L_r,$$

each L_i an ideal in $I^\perp$. But $L_i \subseteq I^\perp$ so $[I, L_i] = 0$ so ideal in L. //

$\Longleftarrow$ Suppose $L = L_1 \oplus \cdots \oplus L_r$, L_i simple ideal. Must show $\operatorname{rad} L = 0$.

$$[\operatorname{rad} L, L_i] \subseteq \operatorname{Rad} L \cap L_i \text{ is a solvable ideal of } L_i \text{ so } = 0.$$

Thus $\operatorname{rad} L \subseteq Z(L) = Z(L_1) \oplus \cdots \oplus Z(L_r) = 0$. //

<u>Cor</u> L semisimple $\Longrightarrow$ L/I is semisimple.

<u>Proof</u> $L = I \oplus I^\perp$ so $L/I \cong I^\perp$. //