Lecture 14 Representations of $\mathfrak{sl}(2,\mathbb{C})$

Recall $e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, $h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ $\quad [e,f] = h$, $[h,e] = 2e$, $[h,f] = -2f$

Goal Construct all irreducible Finite-Dimensional $\mathfrak{sl}(2,\mathbb{C})$ modules. We will see there is one of each dimension

Method 1. Write down answer!

2. Show any irreducible is on list.

- Key is to analyze eigenvectors of h and how they are acted on by e & f, we will see $\varphi(h)$ is diagonalizable always

Answer Let $\mathbb{C}[x,y]$ = polynomials in two variables

V_d = subspace of homogeneous poly of degree d

$V_0 = \mathbb{C}\cdot 1 \qquad V_2 = \langle x^2, xy, y^2 \rangle$

$V_1 = \langle x, y \rangle \qquad V_3 = \langle x^3, x^2y, xy^2, y^3 \rangle$ etc. $\quad \dim V_d = d+1$

Claim We can give V_d an $\mathfrak{sl}(2,\mathbb{C})$ module structure via:

$$\varphi(e) = x\cdot \frac{\partial}{\partial y}$$
$$\varphi(f) = y\cdot \frac{\partial}{\partial x}$$

$$\varphi(h) = x\cdot \frac{\partial}{\partial x} - y\frac{\partial}{\partial y}$$

Check: $[\varphi(e), \varphi(f)] = \varphi(h)$

$[\varphi(h), \varphi(e)] = 2\varphi(e)$ $\qquad$ since actions

$[\varphi(h), \varphi(f)] = -2\varphi(f)$

Ex $[\varphi(e),\varphi(f)]\, x^i y^{d-i} = \varphi(e)\varphi(f)\, x^i y^{d-i} - \varphi(f)\varphi(e)\, x^i y^{d-i}$

$$= \varphi(e)\, i x^{i-1} y^{d-i+1} - \varphi(f)\,(d-i)\, x^{i+1} y^{d-i-1}$$
$$= (d-i+1)\, i\, x^i y^{d-i} - (i+1)(d-i)\, x^i y^{d-i}$$
$$= (id - i^2 + i - id + i^2 - d + i)\, x^i y^{d-i}$$
$$= (2i-d)\, x^i y^{d-i}$$

$$\varphi(h)\, x^i y^{d-i} = i\, x^i y^{d-i} - (d-i)\, x^i y^{d-i} = (2i-d)\, x^i y^{d-i}$$

etc...

Matrices

$$\varphi(e) = \begin{pmatrix} 0 & d & 0 & 0 & & 0 \\ & 0 & 2 & 0 & & 0 \\ & & 0 & 3 & & \vdots \\ & & & \ddots & & \\ & & & & 0 & d \\ & & & & & 0 \end{pmatrix} \qquad \varphi(f) = \begin{pmatrix} 0 & 0 & & & & \\ d & 0 & & & & \\ 0 & d-1 & 0 & & & \\ 0 & 0 & d-2 & \ddots & & \\ 0 & \cdots & & 0 & 1 & 0 \end{pmatrix}$$

$$\varphi(h) = \begin{pmatrix} d & & & & \\ & d-2 & & & 0 \\ & & d-4 & & \\ & 0 & & \ddots & \\ & & & & -d = d-2d \end{pmatrix}$$

Rmk 1. $V_1 = \langle x, y \rangle$ is just the natural module $e \to \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $f \to \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, $h \to \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

2. $V_d \cong Sym^d(V_1)$ for those who know

3. $\varphi(h)$ is diagonalizable w/ distinct e-values $d, d-2, d-4, \dots, -d$

4. Clearly any of the std basis vectors generate V_d, this does not automatically imply V_d is irreducible.

Thm V_d is irreducible.

Proof Let $U \leq V$ be a submodule. Over $\mathbb{C}$ so $\rho(h)|_U$ has an eigenvector, which must be a multiple of some std basis vector.

Thus $x^i y^{d-i} \in U \Rightarrow U = V$. //

Step 2 Suppose V an $sl(2,\mathbb{C})$ module, and $v \in V$ is an eigenvector for h with e-value λ.

Lemma

1. Either $e \cdot v = 0$ or $e \cdot v$ is an e-vector for h w/ evalue $\lambda + 2$
2. Either $f \cdot v = 0$ or $f \cdot v$ is an e-vector for h w/ evalue $\lambda - 2$

Proof $h \cdot e \cdot v = e \cdot h \cdot v + [h,e] \cdot v$

$= \lambda e \cdot v + 2e \cdot v = (2+\lambda) e \cdot v$, similarly for $f \cdot v$ //

Lemma Suppose V is a f.d. $sl(2,\mathbb{C})$ module. Then V contains an e-vector w for h such that $e \cdot w = 0$.

Proof $h: V \to V$ has an e-vector v w/ value λ ($F = \mathbb{C}$). But then

$\{v, e \cdot v, e^2 \cdot v, \ldots\}$ have eigenvalues $\lambda, \lambda+2, \lambda+4, \ldots$

If nonzero, they are linearly independent but $\dim V < \infty$. //

Analyze arbitrary irreducible V of finite dimension

1. Choose $w \in V$ as above so:

$$h \cdot w = \lambda w$$
$$e \cdot w = 0.$$

Consider $\{w, f \cdot w, f^2 \cdot w, \ldots, f^d \cdot w\}$ with $f^d \cdot w \neq 0$, $f^{d+1} \cdot w = 0$

by same proof as above.

Claim $\{w, f\cdot w, f^2\cdot w, .., f^d\cdot w\}$ are a basis of a submodule (hence of V)

Proof Lin ind since all evectors of h w/ distinct e-values

Lemma $e\cdot f^k w \in \mathrm{Span}\{w, f\cdot w, f^2\cdot w, .., f^{k-1}\cdot w\}$

Pf By induction, $k=0$ since $e\cdot w = 0$.

$$e\cdot f^k\cdot w = ef\cdot f^{k-1}w = (fe + [e,h])\, f^{k-1}\cdot w$$
$$= fef^{k-1}w + hf^{k-1}w$$
$$= f\cdot\{\text{span } f^{k-2}\cdot w, f^{k-3}\cdot w ..\} + \ast f^{k-1}w \quad //$$

Claim $\lambda = d$

Proof Matrix of h is $\begin{pmatrix} \lambda & & & \\ & \lambda-2 & & 0 \\ & 0 & \ddots & \\ & & & \lambda-d \end{pmatrix}$ trace $(d+1)\lambda - d(d+1)$

But $h = [e,h]$ so trace $=0$ so $\lambda = d$ //

Claim $V \cong V_d$

Proof Note $\{w, f\cdot w, f^2\cdot w, .., f^d\cdot w\}$ basis of V
$\{x^d, f\cdot x^d, f^2\cdot x^d, .., f^d\cdot x^d\}$ basis of V_d same h-evalues

Define $\varphi(f^k\cdot w) = f^k\cdot x^d \quad V \to V_d$

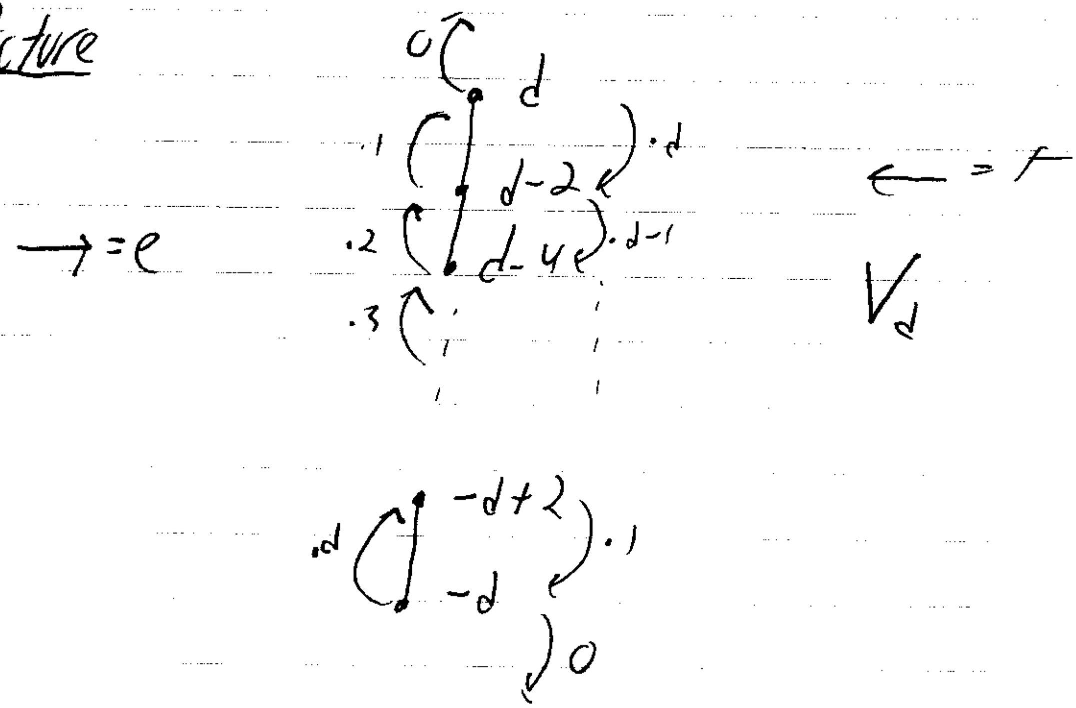
ETS $\varphi(e\cdot f^k\cdot w) = ef^k\cdot x^d$, check by induction. //

Cor Suppose V is an $\mathfrak{sl}(2,\mathbb{C})$ module, $w \in V$ an e-vector for h such that $e \cdot w = 0$. Then

1. $h \cdot w = dw$, some $d = 0, 1, 2, \ldots$
2. Submodule gen by w is $\cong V_d$

Rmks Such a V is called a *highest weight vector*.
d is called a *highest weight*.

Picture



For thought: Carefully review proof to see what, if anything, goes wrong in alg closed of char p