# Lecture 10

**Review** $A \subseteq gl(V)$, $\lambda \in A^*$ is a *weight* of $A$ if $V_\lambda = \{v \in V \mid a(v) = \lambda(a)v \;\; \forall a \in A\}$ is nonzero. Say $V_\lambda$ is the *$\lambda$-weight space*

**Invariance Lemma** Suppose char $F = 0$, $L \subseteq gl(V)$ a subalgebra, $A \subseteq L$ an ideal, and $\lambda \in A^*$ a weight. Then $V_\lambda$ is $L$-invariant.

**Proof** For $y \in L$, $w \in V_\lambda$, we need $y(w) \in V_\lambda$. Let $U = \text{span}\{w, y(w), y^2(w), \ldots\}$

**Lemma** a. We can choose basis of $U$ of form $\{w, y(w), y^2(w), \ldots, y^{m-1}(w)\}$

b. In this basis, any $z \in A$ maps $U$ to itself w/ matrix

$$\begin{pmatrix} \lambda(z) & & * \\ & \lambda(z) & \\ 0 & & \ddots \;\; \lambda(z) \end{pmatrix}$$

**Pf** (a) is clear, $m$ is smallest so $\{w, y(w), \ldots, y^m(w)\}$ is lin dep.

(b) By induction on column #. $z(w) = \lambda(z)\,w$ since $z \in A$, $w \in V_\lambda$. ✓

Inductive Step: $z(y^r(w)) = zy(y^{r-1}(w)) = (yz + [z,y])y^{r-1}(w)$

$= yzy^{r-1}(w) + [z,y]y^{r-1}(w)$ $\in \text{span}\{w, yw, \ldots, y^{r-1}w\}$ by induction since $[z,w] \in A$.

$= y(\lambda(z)y^{r-1}(w) + ?y^{r-2}(w) + \ldots)$

$= \lambda(z)y^r(w) + ?y^{r-1}(w) + \ldots$

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Choose $z = [a,y]$ for $a \in A$. Then $U$ is invariant under $z$ and $a$ by Lemma, under $y$ by const.

Thus $\text{trace}(z) = m \cdot \lambda(z) = tr(ay - ya) = 0$, so

$$\lambda[a,y] = 0 \quad (*)$$

Finally $ay(w) = (ya + [a,y])w$
$= y\lambda(a)w + \lambda([a,y])w = \lambda(a)yw + 0$ so $yw \in V_\lambda$ //

Motivating Thm Suppose $\dim V = n$ and $x \in gl(V)$ is nilpotent. Then $\exists$ a basis of $V$ in which $[x]_B$ is strictly upper $\Delta$.

Proof $x^n = 0$ so $\exists\, v_1 \in \operatorname{Ker} x$. Let $U = \langle v_1 \rangle$. Consider
$\bar{x} : V/U \to V/U$
$\bar{x}(v+U) = xv + U$, well-defined since $U \subset \operatorname{Ker} x$

Then $\bar{x}$ is nilpotent so by induction $\exists$ basis $\{v_2 + U, v_3 + U, \dots, v_n + U\}$ where $\bar{x}$ is str. upper $\Delta$.
Check: $\{v_1, v_2, \dots, v_n\}$ is our desired basis. //

Engel's Thm Suppose $L \subseteq gl(V)$ and every element of $L$ is nilpotent. Then $\exists$ a basis of $V$ so every $x \in L$ is strictly upper triangular.

Step 1 $\exists\, v \in V$ so $xv = 0$ $\forall x \in L$.

Proof By induction on $\dim L$. If $L = \langle x \rangle$ use motivating thm above

Lemma Let $A$ be a maximal subalgebra of $L$. Then $A$ is an ideal of codim $1$.

Proof Let $\bar{L} = L/A$, define $\varphi : A \to gl(\bar{L})$ by

$$\varphi(a)(x + A) = [a,x] + A$$

- $\varphi$ is well-defined since $A$ is a subalgebra

- $[\varphi(a), \varphi(b)](x+A) = \varphi(a)([b,x]+A) - \varphi(b)([a,x]+A))$
$$= [a,[b,x]]+A - [b[a,x]] + A$$
$$= [[a,b],x]+A = \varphi([a,b])(x+A) \text{ so homom}$$

- $a$ nilpotent $\Rightarrow$ $\operatorname{ad} a$ is nilpotent $\Rightarrow$ $\varphi$ is nilpotent

- Thus $\varphi(A)$ is a subalgebra of $gl(\bar{L})$, every element is nilpotent, and $\dim(\varphi(A)) \le \dim A < \dim L$.

By induction $\exists\ y+A \in \bar{L}$ so $\varphi(a)(y+A) = [a,y]+A = 0\ \forall a \in A$.

Thus $[a,y] \in A\ \forall A$. Thus $\tilde{A} = A \oplus \operatorname{span}(y)$ is a subalgebra, thus
$$\tilde{A} = A + \operatorname{span}(y) = L \Rightarrow A \text{ an ideal of codim } 1 \quad //$$

Now apply induction to $A \le gl(V)$. Thus $\exists\ 0 \ne w \in V$ so $a(w) = 0\ \forall\ a \in A$.

$$0 \ne W = \{v \in V \mid a(v) = 0\ \forall a \in A\}$$

By invariance lemma $W$ is $L$-invariant, thus $y(W) \le W$.

Since $y$ is nilpotent, so is $y|_W$. Thus $\exists\ 0 \ne v \in W$ with $yv = 0$.

$v$ is a vector! // End of Step 1

Step 2 Like in Motivating Thm, induct on dim $V$.

Choose $v_1 \in V$ so $xv_1 = 0$ $\forall x \in L$. Let $U = \langle x \rangle$, $\bar{V} = V/U$

$$L \to gl(\bar{V})$$

$x \to \bar{x}$ is a Lie alg homo., all $\bar{x}$ nilpotent

$Im(L)$ subalg of $gl(\bar{V})$ and $\dim(\bar{V}) < \dim V$.

Choose basis $\{v_2+U, v_3+U, \ldots, v_n+U\}$ so all $\bar{x}$ are strict upper $\Delta$.

Then $\{v_1, v_2, \ldots, v_n\}$ works

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Second Version

Before $x \in gl(V)$ nilpotent $\Rightarrow$ ad $x$ nilpotent.

Thm A Lie algebra $L$ is nilpotent $\leftrightarrow$ $\forall x \in L$, $x$ is ad-nilpotent

Proof $\Rightarrow$ Done before

$\Leftarrow$ ad: $L \to gl(L)$, by assumption each ad $x$ is nilpotent

By Engel, $\exists$ basis of $L$ so every ad $x$ is strictly upper $\Delta$

Thus ad $L$ is nilpotent.

By HW $L$ is nilp.