Name: SOLUTIONS

Math 461/561 Midterm Exam - October 20, 2015

1. **(20 points)** Complete the following:

a. Let L be a Lie algebra. The *derived subalgebra* L' is $\cdots$

$$L' = \text{span}\{[x,y] \mid x,y \in L\}$$

b. For L a Lie algebra the *center* $Z(L)$ is $\cdots$

$$Z(L) = \{x \in L \mid [x,y] = 0 \ \forall y \in L\}$$

c. Let V be a module for a Lie algebra L. Say V is *indecomposable* if $\cdots$

there do not exist submodules $V_1, V_2 \neq 0$ with $V \cong V_1 \oplus V_2$

d. A Lie algebra L is *nilpotent* if $\cdots$ there exists n s.t. any n-fold bracket $[x_1,[x_2,[x_3,\ldots[x_{n-1},x_n]]]\ldots] = 0$.

(Equivalently define $L^1 = [L,L]$, $L^k = [L,L^{k-1}]$ then L is nilpotent if $L^n = 0$ for some n.

2. **(20 points)** True or false. If false, give a counterexample.

F a. The image of a Lie algebra homomorphism is an ideal.

ad $sl(2,\mathbb{C}) \subseteq gl(3,\mathbb{C})$ is not an ideal

F b. Suppose L is a solvable Lie subalgebra of $gl(V)$. Then every element of L is upper triangular.

$L = \text{span}\left\{ \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \right\}$

T c. The normalizer $N_L(A)$ of a subalgebra A in L is the largest subalgebra of L in which A is an ideal.

T d. Every nilpotent Lie algebra is solvable.

F e. Lie's theorem holds over an algebraically closed field of characteristic p.

see HW

F f. The adjoint representation of $sl(2, F)$ is faithful for any field.

If char $F=2$ then $h=\begin{pmatrix}1&0\\0&1\end{pmatrix} \in Z(sl(2,F))$

3. **(15 points)** Let L be a dimension three Lie algebra and suppose $L = [L, L]$. Prove (from scratch, not using the classification we did in class!) that L is simple. Hint: Consider homomorphic images of L.

Suppose $\varphi(L) \cong L/\mathrm{Ker}\,\varphi$. Then since $[\varphi(x), \varphi(y)] = \varphi([x,y])$

we easily see $\varphi(L)$ also satisfies the property

that it is its own derived algebra

If $\varphi(L)$ is 1-dim'l it is abelian, so not = to $\varphi(L)$!

~~We see~~ If $\tilde{L} = \langle x, y \rangle$ is 2-dim'l then

$[\tilde{L}, \tilde{L}] = \mathrm{span}\,[x,y]$ is 1-dim'l,

Thus L has no 1 or 2-dim'l homomorphic images!

Thus no ideals of dim >0 since L/I is

a homom. image

4. **(15 points)** Let L be a Lie algebra. Suppose I is an ideal of L such that the quotient L/I is nilpotent and such that for each $x \in L$, the map $\operatorname{ad} x : I \to I$ is nilpotent. Prove that L is nilpotent.

By Engel's Thm L is nilpotent $\Leftrightarrow$ ad x is nilp $\forall x \in L$.

So L/I nilpotent implies $(\operatorname{ad} x)^n(L) \subseteq I$ for some n.
for $x \in L$

But by assumption $\exists\, m$ so $(\operatorname{ad} x)^m : I \to I$ is 0.

Thus $(\operatorname{ad} x)^{n+m}$ is the 0 map on L

Thus $\operatorname{ad} x : L \to L$ is nilpotent

So by Engel, L is nilpotent.

5. **(10 points)** Suppose V is an $sl(2, \mathbb{C})$ module. Suppose further there is $0 \neq v$ such that $h \cdot v = \lambda v$. Prove that $e \cdot v$ is either 0 or an eigenvector of h with eigenvalue $\lambda + 2$.

$h \cdot (e \cdot v) = ([h, e] + e \cdot h) v$ by def of V being an L-module

$= 2e \cdot v + e \cdot h \cdot v$

$= 2e \cdot v + \lambda e \cdot v = (2 + \lambda) e \cdot v$

So if $e \cdot v \neq 0$ then it is an e-vector of h w/ evalue $2 + \lambda$.

6. **(10 points)** Let $V = \oplus_{i=1}^{r} S_i$ be an $sl(2,\mathbb{C})$ module where each S_i is irreducible. Let W_0 be the 0 eigenspace for h, i.e. $W_0 = \{v \in V \mid h \cdot v = 0.\}$ Similarly let W_1 be the 1 eigenspace. Prove that $r = \dim W_0 + \dim W_1$. Hint: What are the eigenvalues of h in the irreducible representation V_d?

First note that the eigenspaces of V are just direct sums of e-spaces of each S_i w/ the same eigenvalue.

If d is even then h has e-values

$d, d-2, \ldots, 2, 0, -2, \ldots, -d$ on V_d

If d is odd then h has e-values

$d, d-2, \ldots, 1, -1, \ldots, -d$ on V_d

So each S_i contributes a 1-dim'l e-space of e-value 0 or a 1-dim'l of e-value 1

Thus $r = \dim W_0 + \dim W_1$

7. **(10 points)** Suppose L is a Lie algebra, over $\mathbb{C}$ such that $Z(L) \cap L' \neq 0$. Prove that L has no faithful irreducible representations.

Let $0 \neq z \in Z(L) \cap L'$. Let S be irred

Then by Schur's Lemma z acts

as a scalar on S, i.e.

$$\varphi_S(z) = \lambda \, Id$$

But $z \in L'$ so trace $\varphi_S(z) = 0$

$$\Rightarrow \lambda \cdot \dim S = 0$$

$$\Rightarrow \lambda = 0.$$

Thus $z \in \ker \varphi_S$ so not faithful