Name: SOLUTIONS

Math 353 Midterm Exam - October 14, 2016

Instructions: You may not use any notes, books, calculators, etc... It is ok if your final answers include binomial coefficients.

1. **(50 points)** Short answer, little or partial credit.

a. Write down the standard Young tableaux of shape $\lambda = (3, 2)$.

$$\begin{matrix}1&2&3\\4&5\end{matrix} \qquad \begin{matrix}1&2&4\\3&5\end{matrix} \qquad \begin{matrix}1&2&5\\3&4\end{matrix} \qquad \begin{matrix}1&3&4\\2&5\end{matrix} \qquad \begin{matrix}1&3&5\\2&4\end{matrix}$$

b. A class of 20 students wishes to elect a president, 3 senators and 3 representatives (all different students). How many ways are there to do this?

$$20 \cdot \binom{19}{3}\binom{16}{3}$$

c. In how many ways can 20 identical balls be placed in 4 distinct cups so that each cup contains an odd number of balls?

$$2x+1+2y+1+2z+1+2w+1=20 \qquad x,y,z,w \geq 0$$

$$x+y+z+w=8$$

"8 dots" 3 pluses

$$\binom{11}{3}$$

d. State the recurrence relation satisfied by the binomial coefficients $C(n,k)$. Also find $\sum_{k=0}^{n} C(n,k)$.

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$$

$$\sum_{k=0}^{n} \binom{n}{k} = 2^n$$

e. Calculate the Stirling number $S(6,3)$.

Recall $S(n,1) = S(n,n) = 1$

$S(n,k) = S(n-1,k-1) + k\,S(n-1,k)$

	1	2	3	4	5	6
$n=1$	1					
2	1	1				
3	1	3	1			
4	1	7	6	1	1	
5	1	15	25	10		
6	1	31	90			1

90

f. What is counted by the Stirling number of the first kind $s(n, k)$?

# of permutations in S_n
with exactly k cycles

g. How many partitions of 8 are there with exactly 3 parts? State a "balls in boxes" problem that this is a solution to.

611
521
431
422
3~~2~~32

total of 5

8 identical balls
3 identical boxes
no empties

h. A five-card poker hand is dealt. What is the probability of getting a full house (e.g. AAAKK, 3 of a kind and a pair).

$$\frac{13\cdot\binom{4}{3}\cdot 12\cdot\binom{4}{2}}{\binom{52}{5}}$$

i. Consider the permutation 351462 in one-line notation. Find the corresponding pair (P, Q) of standard tableaux under the Robinson-Schensted algorithm.

$$\begin{array}{llll} 1 & \cancel{3}\,\cancel{5}\,\cancel{4}\ 2 & 6 \\ 3 & 4\,\cancel{5} & \\ 5 & & \end{array} \qquad \begin{array}{lll} 1 & 2 & 5 \\ 3 & 4 & \\ 6 & & \end{array}$$

$$\left(\begin{array}{lll} 1 & 2 & 6 \\ 3 & 4 & \\ 5 & & \end{array} \ , \ \begin{array}{lll} 1 & 2 & 5 \\ 3 & 4 & \\ 6 & & \end{array} \right)$$

j. Write down the Schur polynomial $s_{(2,1)}(x_1, x_2, x_3)$.

$$\begin{array}{ll} 1 & 1 \\ 2 & \end{array} \quad \begin{array}{ll} 1 & 1 \\ 3 & \end{array} \quad \begin{array}{ll} 1 & 2 \\ 2 & \end{array} \quad \begin{array}{ll} 1 & 2 \\ 3 & \end{array} \quad \begin{array}{ll} 1 & 3 \\ 2 & \end{array} \quad \begin{array}{ll} 2 & 2 \\ 3 & \end{array} \quad \begin{array}{ll} 2 & 3 \\ 3 & \end{array} \quad \begin{array}{ll} 1 & 3 \\ 3 & \end{array}$$

$$x_1^2x_2 + x_1^2x_3 + x_1x_2^2 + 2x_1x_2x_3 + x_2^2x_3 + x_2x_3^2 + x_1x_3^2$$

2. **(15 points)** Recall that a bridge hand contains 13 cards and there are 52 cards in the deck so there are a total of $\binom{52}{13}$ bridge hands. A hand is said to contain a void if it does not have cards from all four suits. Calculate the probability that a random bridge hand contains a void.

$$\# V_H = V_D = V_C = V_S = \binom{39}{13}$$

$$\# V_H \cap V_D = \binom{26}{13}$$

$$\# V_H \cap V_D \cap V_S = \binom{13}{13}$$

By I/E # hands w/ a void is

$$|V_H \cup V_D \cup V_C \cup V_S| = 4 \cdot \binom{39}{13} - \binom{4}{2}\binom{26}{13} + \binom{4}{3}\binom{13}{13}$$

$$\boxed{\text{Prob} = \frac{4\binom{39}{13} - 6\binom{26}{13} + 4\binom{13}{13}}{\binom{52}{13}}}$$

3. **(15 points)** Recall that $p_k(n)$ counts the number of partitions of n with less than or equal to k parts. Give a combinatorial proof that

$$p_k(n) = p_{k-1}(n) + p_k(n-k).$$

Let X = all partitions of n with $\leq K$ parts.

Let $A \subseteq X$ be those with exactly K parts

Let $B \subseteq X$ be those with $< K$ parts. So $\#X = \#A + \#B$

Now $\#B = p_{K-1}(n)$ is clear, B is all partitions of n with $\leq K-1$ parts

For each $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_K) \in A$, removing the first column from Ferrers diag. gives

$\bar{\lambda} = (\lambda_1 - 1, \lambda_2 - 1, \ldots, \lambda_{K-1})$ a partition of $n-K$ w/ $\leq K$ parts

Adding a 1^{st} column of length K undoes this so we have a bijection proving $\# A = p_K(n-k)$!

Thus $p_K(n) = p_K(n-k) + p_{K-1}(n)$. //

4. **(20 points)** a. Define the Catalan number C_n.

$$\frac{\binom{2n}{n}}{n+1}$$

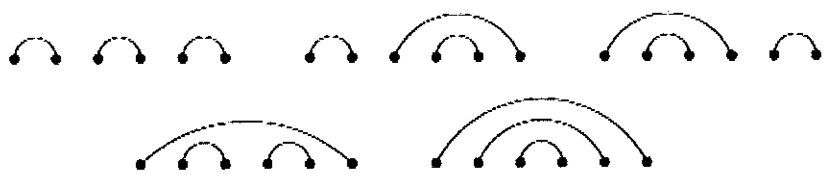
b. State two things that are counted by C_n.

- Dyck paths $(0,0)$ to (n,n)
- triangulations of $n+2$ gon
- 231 avoiding permutations in S_n
- expressions of length $n+1$

c. State the recursion satisfied by the Catalan numbers.

$$C_{n+1} = \sum_{k=0}^{n} C_k C_{n-k}$$

61. ***Noncrossing (complete) matchings*** on $2n$ vertices, i.e., ways of connecting $2n$ points in the plane lying on a horizontal line by n nonintersecting arcs, each arc connecting two of the points and lying above the points.

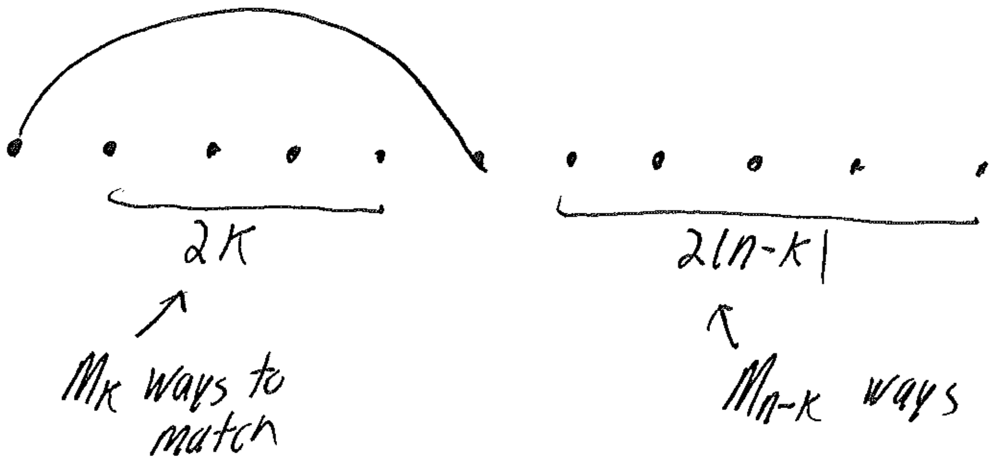


d.

Let M_n be the number of noncrossing complete matchings on $2n$ vertices (defined above from Stanley's book). The diagram above shows that $M_3 = 5$. Prove that $M_n = C_n$.

Calculate M_{n+1}.

Group matchings by # of dots under the arc containing left dot. There must be an even # $2k$ since arcs cannot cross.



$0 \leq k \leq n$.

Thus $M_{n+1} = \sum_{k=0}^{n} M_k M_{n-k}$.

Clearly $M_1 = 1$ 

So by HW, $M_n = C_n$. //