Math 353 Homework #8- Due Wednesday 11/2/16 SOLUTIONS

1. $\sigma\tau = (1, 3, 5, 4, 2)(6, 8, 10, 12, 7)(9, 11)$, $\tau\sigma = (1, 3, 2, 4, 6)(5, 7, 9, 11, 8)(10, 12)$.
$\tau^{-1} = (1, 4, 3, 2)(5, 7, 6)(8, 12, 11, 10, 9)$
$\tau\sigma\tau^{-1} = (2, 3, 4)(1, 6, 7)(5, 9)(10, 11, 12, 8)$.
The order of an n cycle is n. The permutation σ has order 12 and τ has order 60.

In one-line notation we have $\sigma = 2, 3, 1, 5, 6, 4, 8, 7, 10, 11, 12, 9$ which is not 231-avoiding.

2. There are 4 elements in our group so the maximum order of an element is 4. If g has order 4 then $\{e, g, g^2, g^3\}$ are all distinct so our group G is cyclic of order 4. Suppose we have an element x of order 3, so we can write $G = \{e, x, x^2, y\}$. The submatrix of the Cayley table from $\{e, x, x^2\}$ already has a e, x, x^2 in each row and column, so there is no way to fill in the row for y and no such group exists. (Or use Lagrange's theorem to rule out this case). Finally we come to the case where all nonidentity elements have order 2 so let $x \neq y$ have order two. Then xy is not equal to x or y by cancellation so it also has order two. Thus we have $G = \{e, x, y, xy\}$ with $x^2 = y^2 = (xy)^2 = e$. Use this equation to show $xy = yx$ so we have the Klein 4 group.

3. (11.3.2B) G has only 4 elements of order 1 or 2 so any hypothetical subgroup must contain at least one 3-cycle, and it's inverse. However it is easy to check that $\{e, (12)(34), (13)(24), (14)(23), (abc), (acb)\}$ is not closed under multiplication. So we have at least two different pairs of 3cycles, say (a, b, c) and (a, b, d) without loss of generality. Multiplying these in both orders gives $(ac)(bd)$ and $(ad)(bc)$ so we end up with already 4 cycles, 2 elements of order 2 and the identity. Too big! Thus G has no subgroup of order 6.

4. (11.4.1B) For any of the 4 corners of a tetrahedron one can fix that corner and rotate the opposite triangle by 120 or 240 degrees, so this gives 8 symmetries of order 3. There are also 3 rotations of 180 degrees which have order 2. Allowing orientation reversing you also get reflections of order 2 for a total of 24 symmetries.

5. (11.5.3B) To find an element of maximal order in S_n we need to find the partition $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_t)$ of n that maximizes the gcd of $\{\lambda_1, \lambda_2, \dots, \lambda_t\}$. Then any permutation of that cycle type will work.

n	sigma	order
1	e	1
2	(12)	2
3	(123)	3
4	(1234)	4
5	(123)(45)	6
6	(123)(45)	6
7	(1234)(567)	12
8	(12345)(678)	15
9	(12345)(6789)	20
10	(12345)(678)(9,10)	30

6. The following elements have order 12:$\{1, 5, 7, 11\}$.

The following elements have order 6: $\{2, 10\}$

The following elements have order 4: $\{3, 9\}$.

The following elements have order 3: $\{4, 8\}$.

The following elements have order 2: $\{6\}$.
And $\{0\}$ has order 1.

7. The matrix of a 120 degree rotation has order 3, so for example $\begin{pmatrix} -1/2 & \sqrt{3}/2 \\ -\sqrt{3}/2 & -1/2 \end{pmatrix}$. The matrix $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ has the property that $A^n = \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}$, so clearly has infinite order.

8. Let $z_1, z_2 \in Z(G)$ and let $g \in G$. Then:

$$\begin{aligned} (z_1 z_2)g &= z_1(g z_2) \text{ by associativity and since } z_2 \text{ is in the center.} \\ &= g(z_1 z_2) \text{ by associativity and since } z_1 \text{ is in the center.} \end{aligned}$$

Thus $z_1 z_2 \in Z(G)$ so $Z(G)$ is closed under the operation. Now suppose $z \in Z(G)$ and let $g \in G$. Then since z is in the center we get:

$$zg^{-1} = g^{-1}z.$$

Inverting both sides of this equation gives us:

$$gz^{-1} = z^{-1}g$$

so z^{-1} is in the center. Thus $Z(G) \leq G$.