Math 353 Homework #7- Due Wednesday 10/26/14

1. 7.2.1B (i) By definition we have: $F_k(x) = \sum n^k x^n$ so taking the derivative and multiplying by x we obtain:

$$xF_k'(x) = x\sum n^{k+1}x^{n-1} = \sum n^{k+1}x^n$$

and this is equal to $F_{k+1}(x)$ by definition.

(ii) We prove by induction that

$$F_k(x) = \frac{P_k(x)}{(1-x)^{k+1}}$$

where P_k is monic of degree k. The $k = 0, 1$ cases were done in class. So assume the formula holds for $F_k(x)$. Thus:

$$F_k(x) = \frac{x^k + \cdots}{(1-x)^{k+1}}.$$

Now use part (i) and the quotient rule:

$$\begin{aligned} F_{k+1}(x) &= xF_k'(x) \\ &= x\frac{(1-x)^{k+1}(kx^k + \cdots) - (x^k + \cdots)(-(k+1)(1-x)^k)}{(1-x)^{2k+2}} \\ &= x\frac{(1-x)(kx^k + \cdots) - (x^k + \cdots)(-(k+1))}{(1-x)^{k+2}} \end{aligned}$$

Check that the coefficicent of x^{k+1} in the numerator is $-k+k+1 = 1$ and that this is the highest degree term so we have

$$F_{k+1}(x) = \frac{x^{k+1} + \cdots}{(1-x)^{k+2}}$$

as desired.

2. 7.2.2B Let a_n be the sum of the first n cubes. So $a_0 = 0$ and

$$a_{n+1} = a_n + (n+1)^3 = a_n + n^3 + 3n^2 + 3n + 1.$$

Hence:

$$(1)\quad \sum_{n=0}^{\infty} a_{n+1}x^n = \sum_{n=0}^{\infty} a_n x^n + \sum_{n=0}^{\infty} n^3x^n + 3\sum_{n=0}^{\infty} n^2x^n + 3\sum_{n=0}^{\infty} nx^n + \sum_{n=0}^{\infty} x^n.$$

Let $f(x)$ be the generating function desired. So the LHS of (1) is $\frac{1}{x}f(x)$. Using our knowledge of the generating functions for the sequences $\{1\}$, $\{n\}$, $\{n^2\}$ and $\{n^3\}$ we obtain:

$$\frac{1}{x}f(x) = f(x) + \frac{x(1+4x+x^2)}{(1-x)^4} + \frac{3x(1+x)}{(1-x)^3} + \frac{3x}{(1-x)^2} + \frac{1}{1-x}.$$

Solving for $f(x)$ gives:

$$f(x) = \frac{x(x^2+4x+1)}{(1-x)^5}.$$

3. 7.5.2B

If our sequence starts with a consonant (21 choices) the remaining n-1 terms are arbitrary (as long as they follow the rule). If it starts with a vowel (5 choices) the next one must be a consonant (21 choices) and then the remaining $n-2$ are arbitrary. This proves the recursion:

$$a_n = 21a_{n-1} + 105a_{n-2}.$$

$a_1 = 26$ is easy. For length two the only thing outlawed is vowel-vowel, which is 25 choices. So $a_2 = 26^2 - 25 = 651$.

The polynomial is $x^2 - 21x - 105 = 0$. This has roots

$$\frac{21 \pm \sqrt{861}}{2}.$$

So a solution is of the form:

$$a_n = A(\cdot\frac{21+\sqrt{861}}{2})^n + B\cdot(\frac{21-\sqrt{861}}{2})^n.$$

We can solve for A and B using the initial conditions! You should obtain:

$$A = \frac{1}{2} + \frac{31}{1722}\sqrt{861}, B = A = \frac{1}{2} - \frac{31}{1722}\sqrt{861}.$$

4. The "tower of Hanoi" is a puzzle consisting of 3 vertical posts mounted on a board and some number n rings of different diameters. In standard form all rings are stacked on one post in order with the largest ring on the bottom. A solution consists of first choosing a second post on which the rings are to be stacked, then moving the rings from post to post in such a way that a larger ring is never placed on top of a smaller ring. The goal is to get all the rings to the second post. Let a_n be the minimum number of moves to solve a puzzle with n rings.

a. Explain why $a_{n+1} = 2a_n + 1$.

b. Find the number of moves needed for n rings. In particular what if $n = 5$.

Suppose you have pegs A, B, C and you are trying to move the rings from A to B. A little thought shows that a successful solution must first move the top $n-1$ rings over to C, then the biggest one moves from A to B, then the $n-1$ rings must move from C to B. This is clear because the largest ring can only move onto an empty peg. This gives us the recursion $a_n = 2a_{n-1} + 1$. So $a_1 = 1, a_2 = 3, a_3 = 7, a_4 = 15, a_5 = 32$.

It seems these numbers are of the form $a_n = 2^n - 1$. We can easily prove this is the correct formula using induction. Suppose it is true for a_n. Then:

$$a_{n+1} = 2a_n + 1 = 2(2^n - 1) + 1 = 2^{n+1} - 1$$

as desired.
5. 8.1.1B

This generating function is counting partitions of 24 with distinct prime parts. There are 5, namely $(19,5)$, $(19,3,2)$, $(17,7)$, $(17,5,2)$, $(13,11)$.

6. 8.1.3B i. We get $a(n)=b(n)$ and the values are 1,2,2,4,5,7,9,13 for n running 1 to 8.

ii. The generating function for $\{a(n)\}$ is:

$$A(x)=\frac{1}{(1-x)(1-x^2)(1-x^4)(1-x^5)(1-x^7)(1-x^8)\cdots}.$$

The generating function for $\{b(n)\}$ is:

$$B(x)=(1+x+x^2)(1+x^2+x^4)(1+x^3+x^6)(1+x^4+x^8)\cdots .$$

We have the identity

$$1+x^k+x^{2k}=\frac{1-x^{3k}}{1-x^k}.$$