Math 353 Fall 2016 - Homework #4 Solutions

3.3.6B We prove that

$$S(n,4) = \frac{1}{24}(4^n) - \frac{1}{6}(3^n) + \frac{1}{4}(2^n) - \frac{1}{6}$$

by induction on n. The base case $n = 4$ is easy, check that $1 = 256/24 - 81/6 + 16/4 - 1/6$.

Now assume the result is true for $S(k,4)$ and prove it for $S(k+1,4)$.

$$\begin{aligned} S(k+1,4) &= S(k,3) + 4S(k,4) \\ &= \frac{1}{2}(3^{k-1} - 2^k + 1) + 4(\frac{1}{24}(4^k) - \frac{1}{6}(3^k) + \frac{1}{4}(2^k) - \frac{1}{6} \end{aligned}$$

by the known result for $S(k,3)$ and the induction hypothesis. Now just simplify this to get the desired result.

3.3.8B The key observation here is to choose a nondecreasing integer we only must choose the digits, and then there is a unique answer obtained by placing them in nondecreasing order. So to get a k digit nondecreasing integer we must put k identical balls in 9 boxes labeled 1-9. (there can we no zeroes except for the number 0). There are $\binom{8+k}{8}$ ways to do this. For numbers less than 1,000,000 we can have 1, 2, 3, 4, 5 or 6 digits. So the final answer is:

$$\binom{14}{8} + \binom{13}{8} + \binom{12}{8} + \cdots + \binom{9}{8} = 5004$$

total ways.

4.1.5B If two coins are tossed the outcomes HH, HT, TH, TT are all equally likely. So we can think of this problem as sampling with replacement from a set of 4 things. Thus we just need to decide how big s should be for $\Theta(4,s)$ to get > 0.9. It turns out that $s = 13$ is what we need.

4.1.7B Same as above but now we need $> .99$ which requires s to be 21.

4.2.1B If you think about this right we have 4 people and 4 jobs. We are looking to assign them to jobs in a way so each person has a different forbidden job. So this problem is precisely equivalent to counting the number of derangements in S_4, which is 9.

4.3.2A See back of the book

4.3.2B Theorem 4.5 states that:

$$S(n,k) = \frac{1}{k!}\sum_{s=0}^{k-1}(-1)^s C(k,s)(k-s)^n.$$

We want to prove this formula gives us what is in Theorem 3.5, namely that $S(n,1) = S(n,n) = 1$ and $S(n,2) = 2^{n-1} - 1$.

Theorem 4.5 gives $S(n,1) = C(1,0)1^n = 1$.
It also gives: $S(n,2) = \frac{1}{2!}\sum_{s=0}^{1}(-1)^s C(2,s)(2-s)^n$ which is $\frac{1}{2}(2^n - 2) = 2^{n-1} - 1$.
Checking for $S(n,n)$ is much harder.