Lecture 6

Review $\vec{r}(t) = (f(t), g(t), h(t))$ parametrizes a space curve C

$\vec{r}'(t) = (f'(t), g'(t), h'(t))$ is tangent vector, can get tangent line at pt using tang vector as direction

$\vec{T}(t) = \dfrac{\vec{r}'(t)}{|\vec{r}'(t)|}$ unit tangent vector

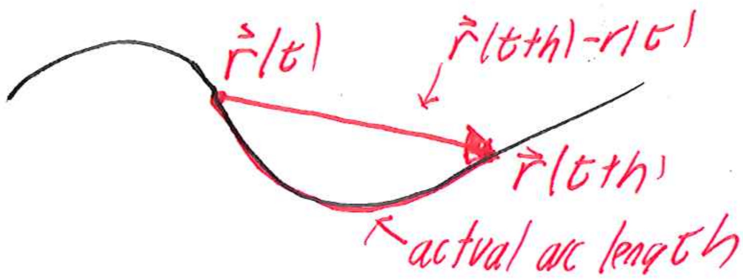
Ex $\vec{r}(t) = (e^t, 2t)$. Sketch, find $\vec{r}'(t)$, Find tang line at $t=1$

Ex (#22) $\vec{r}(t) = \langle e^{2t}, e^{-2t}, te^{2t} \rangle$ Find $\vec{T}(0)$, $\vec{r}''(0)$ and $\vec{r}'(t) \cdot \vec{r}''(t)$

Ex Suppose $|\vec{r}(t)|$ is constant. Prove $\vec{r}'(t) \perp \vec{r}(t)$

Cor If $|\vec{r}'(t)|$ is constant then $\vec{r}'(t) \perp \vec{r}''(t)$ interpret?

Arc length Rmk Given $\vec{r}(t)$ and small h, then $|\vec{r}(t+h) - \vec{r}(t)|$ approximates arc length from $\vec{r}(t)$ to $\vec{r}(t+h)$



Thus approximate speed is

$$\frac{|\vec{r}(t+h) - \vec{r}(t)|}{h}$$

Conclude $|\vec{r}'(t)| = \text{speed}$

$|\vec{r}'(t)|\Delta t \approx \text{distance}$

<u>Thm</u> Suppose $\vec{r}(t)$ parametrizes a curve C traversed once as $a \le t \le b$

Then $$\text{Length } C = \int_a^b |\vec{r}\,'(t)|\,dt$$

<u>Ex 1</u> $\vec{r}(t) = (r\cos t, r\sin t)$ $0 \le t \le 2\pi$ (Circle radius r)

Warning $\int_a^b |\vec{r}\,'(t)|\,dt$ measures distance travelled not length of curve if you traverse more than once

<u>Ex 2</u> $\vec{r}(t) = (x(t), y(t))$ $$\boxed{L = \int_a^b \sqrt{x'(t)^2 + y'(t)^2}\,dt}$$

<u>Ex 3</u> $y = f(x)$ $a \le x \le b$ parametrize as

$\vec{r}(x) = (x, f(x))$ $\vec{r}\,'(x) = (1, f'(x))$ $|\vec{r}\,'(x)| = \sqrt{1 + f'(x)^2}$

$$\boxed{L = \int_a^b \sqrt{1 + f'(x)^2}\,dx}$$

Should be familiar from MTH 142

<u>Ex 4</u> $\vec{r}(t) = (\cos t, \sin t, \ln\cos t)$ $0 \le t \le \pi/4$ Find A.L.

$\vec{r}\,'(t) = (-\sin t, \cos t, \tan t)$ $|\vec{r}\,'(t)| = \sqrt{\sin^2 t + \cos^2 t + \tan^2 t}$

$= \sqrt{1 + \tan^2 t} = \sqrt{\sec^2 t}$

$= \sec t$ since $\sec t > 0$ here

$$A.L. = \int_0^{\pi/4} \sec t\,dt = \ln|\sec t + \tan t|\Big|_0^{\pi/4} \leftarrow \text{tables}$$

$$= \ln|\sqrt{2} + 1| - \ln|1 + 0| = \ln(\sqrt{2} + 1)$$

* Arc Length Integrals are Hard!

Parametrize By Arc Length

Suppose $\vec{r}(t) = (f(t), g(t), h(t))$ parametrizes curve once.

"Arc length Function" $\quad s(t) = \int_a^t |\vec{r}\,'(u)|\,du \quad$ = A.L. from $t=a$ to $t=t$

Sometimes solve $t = t(s)$ and plug in to get $\vec{r} = \vec{r}(t(s))$

parametrized by A.L.
Speed = 1

$$\boxed{\frac{ds}{dt} = |\vec{r}\,'(t)| \text{ by FTOC}}$$

Curvature

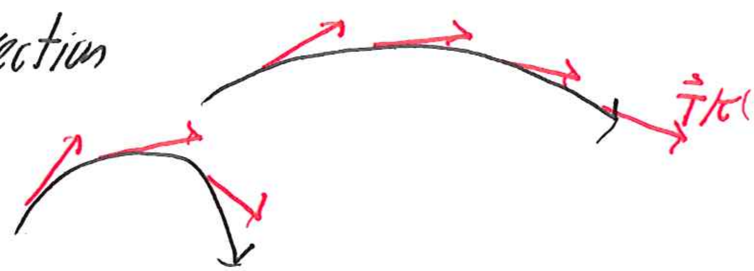
No corners   No stopping

Def $\vec{r}(t)$ is *smooth* if $\vec{r}\,'(t)$ is continuous and $\vec{r}\,'(t) \neq 0$.

Rmk $\vec{r}(t) = (t, t^2)$ is smooth parametrization of curve $y = x^2$

$\vec{r}(t) = (t^3, t^6)$ is not since $\vec{r}\,'(t) = (3t^2, 6t^5)$
not smooth at $t=0$.

Recall $\vec{T}(t) = \dfrac{\vec{r}\,'(t)}{|\vec{r}\,'(t)|}$ gives direction



* How fast does $\vec{T}(t)$ change?
Don't want answer to depend on speed

Def The *curvature* of curve is $\left| \dfrac{d\vec{T}(t)}{ds} \right|$ rate of change of $\vec{T}$ wrt arc length

$$\left|\frac{d\vec{T}(t)}{ds}\right| = \left|\frac{d\vec{T}/dt}{ds/dt}\right| = \frac{|\vec{T}'(t)|}{|r'(t)|}$$

$$\boxed{* \ \kappa(t) = \frac{|\vec{T}'(t)|}{|\vec{r}'(t)|}}$$

* curvature is a scalar quantity * ≥ 0

Key example $\vec{r}(t) = (a\cos t, a\sin t)$ circle radius r.

$$\rightsquigarrow \kappa(t) = 1/a$$

* circle of radius r has constant curvature 1/r

Other Formulas for curvature..

Fact
$$\boxed{\kappa(t) = \frac{|\vec{r}'(t) \times \vec{r}''(t)|}{|\vec{r}'(t)|^3}}$$

accel parallel to velocity means no curvature!

Ex $\vec{r}(t) = (t, t^2, t^3)$

Example $y = f(x)$ parametrize $\vec{r}(x) = (x, f(x))$

$$\vec{r}'(x) = (1, f'(x), 0) \quad \vec{r}''(x) = (0, f''(x), 0)$$

$$r' \times r'' = (0, 0, f''(x)) \quad |\vec{r}(t)| = \sqrt{1 + f'(x)^2}$$

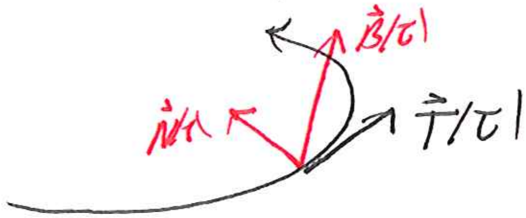
$$\boxed{\kappa(x) = \frac{|f''(x)|}{\left(1 + f'(x)^2\right)^{3/2}}}$$

Frames (on HW, not to lear)

Def Unit normal $\vec{N}(t) = \dfrac{\vec{T}'(t)}{|\vec{T}'(t)|}$

* $|\vec{T}(t)| = 1$ so $\vec{T} \cdot \vec{T}' = 0$

Binormal $\vec{B}(t) = \vec{T}(t) \times \vec{N}(t)$



Frenet-Serret Frame

Def $\vec{T}$ & $\vec{N}$ determine <u>osculating plane</u>

Osculating circle

↓

same normal, tang, curvature.