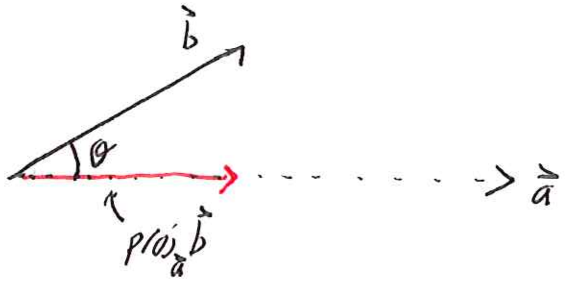
Lecture 3

Review $\vec{a}=(a_1,a_2,\dots,a_n)$ $\vec{b}=(b_1,b_2,\dots,b_n)$ $\vec{a}\cdot\vec{b}=a_1b_1+a_2b_2+\dots+a_nb_n=\sum_{i=1}^{n}a_ib_i$

Fact $\boxed{\vec{a}\cdot\vec{b}=|\vec{a}||\vec{b}|\cos\theta}$ θ is $\measuredangle$ btw $\vec{a}$ & $\vec{b}$

Projection:



1. $\text{comp}_{\vec{a}}\vec{b}=|\vec{b}|\cos\theta=\dfrac{\vec{a}\cdot\vec{b}}{|a|}=\boxed{\vec{b}\cdot\dfrac{\vec{a}}{|\vec{a}|}}$ ← unit vector in direction of $\vec{a}$

2. $\text{proj}_{\vec{a}}\vec{b}=(\text{comp}_{\vec{a}}\vec{b})\cdot\dfrac{\vec{a}}{|\vec{a}|}=\boxed{\dfrac{\vec{a}\cdot\vec{b}}{\vec{a}\cdot\vec{a}}\,\vec{a}}$

Work $=\vec{F}\cdot\vec{D}$

Cross Product

Def $\vec{a}=(a_1,a_2,a_3)$, $\vec{b}=(b_1,b_2,b_3)$

$$\vec{a}\times\vec{b}=(a_2b_3-a_3b_2,\ a_3b_1-a_1b_3,\ a_1b_2-a_2b_1)$$

Example:

Observe

1. $\times:\mathbb{R}^3\times\mathbb{R}^3\to\mathbb{R}^3$, special to 3 dimensions

2. $\vec{a}\times\vec{b}=\begin{vmatrix}\vec{i}&\vec{j}&\vec{k}\\a_1&a_2&a_3\\b_1&b_2&b_3\end{vmatrix}$

3. $\vec{a}\times\vec{b}=-\vec{b}\times\vec{a}$, so in particular $\vec{a}\times\vec{a}=0$

<u>Thm</u> 1. $\vec{a}\times\vec{b}$ is $\perp$ to both $\vec{a}$ and $\vec{b}$, i.e. $\vec{a}\cdot(\vec{a}\times\vec{b}) = \vec{b}\cdot(\vec{a}\times\vec{b}) = 0$

2. $|\vec{a}\times\vec{b}| = |\vec{a}||\vec{b}|\sin\theta$

<u>Cor</u> $\vec{a}\times\vec{b} = \vec{0}$ if & only if $\vec{a}$ and $\vec{b}$ are parallel

<u>Proof</u> 1. Just check!

2. Expand out, $|\vec{a}\times\vec{b}|^2 = (\vec{a}\times\vec{b})\cdot(\vec{a}\times\vec{b})$ 12 terms!

$= |\vec{a}|^2|\vec{b}|^2 - (\vec{a}\cdot\vec{b})^2$

$= |\vec{a}|^2|\vec{b}|^2 - |\vec{a}|^2|\vec{b}|^2\cos^2\theta$

$= |a|^2|b|^2\sin^2\theta$ but $\sin\theta \geq 0$ so take sqr. //

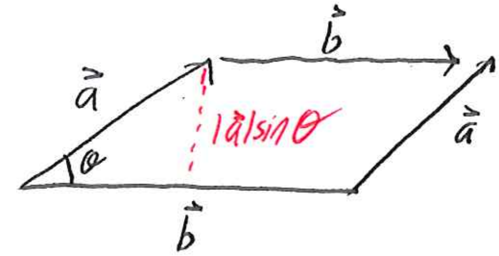
<u>Q</u> Which way does $\vec{a}\times\vec{b}$ point? Two possibilities

<u>A</u> Right hand rule!

Curl Fingers $\vec{a}$ to $\vec{b}$, thumb points toward $\vec{a}\times\vec{b}$.

<u>Ex</u> $\vec{i}\times\vec{j} = \vec{k}$ $\vec{j}\times\vec{i} = -\vec{k}$

<u>Rmk</u>



Area ▱ $= b\cdot h$

$= |\vec{a}||\vec{b}|\sin\theta$

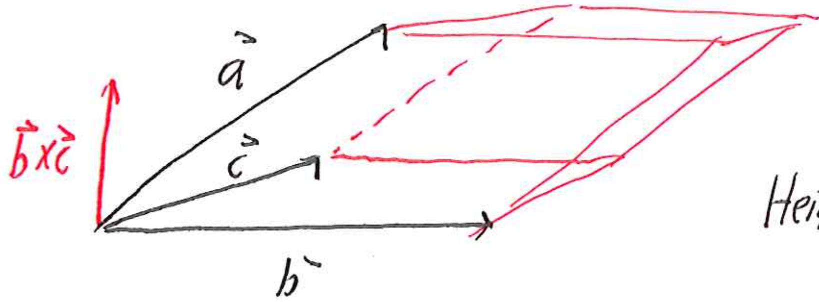
$= |\vec{a}\times\vec{b}|$

So $|\vec{a}\times\vec{b}|$ is area of parallelogram.

<u>Example</u> 1. Find a vector $\perp$ to a plane containing $\overset{P}{(1,1,1)}$, $\overset{Q}{(1,-2,1)}$, $\overset{R}{(-3,1,5)}$

2. Find area of ΔPQR

<u>Scalar Triple Product</u> Problem: Find volume of parallelepided



Area base $= |\vec b \times \vec c|$

Height $=$ comp of $\vec a$ in direction $\vec b \times \vec c$

$$= \vec a \cdot \frac{(\vec b \times \vec c)}{|\vec b \times \vec c|}$$

$$\text{Volume} = (\text{area base}) \cdot H = |\vec a \cdot (\vec b \times \vec c)|$$

<u>Thm</u> $|\underbrace{\vec a \cdot (\vec b \times \vec c)}_{\text{called scalar triple product}}|$ is volume of parallelepiped formed by $\vec a, \vec b, \vec c$

<u>EX</u> 1. Find vol of parallelepided det by $\vec a = \langle 1,2,3 \rangle$ $\vec b = \langle -1,1,2 \rangle$ $\vec c = \langle 2,1,1 \rangle$

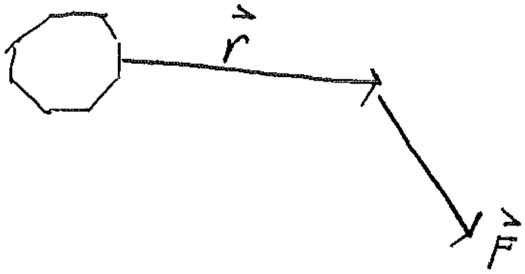
2. Use scalar triple product to verify

$\vec u = i + 5j - 2k$

$v = 3i - j$

$w = 5i + 9j - 4k$ are coplanar

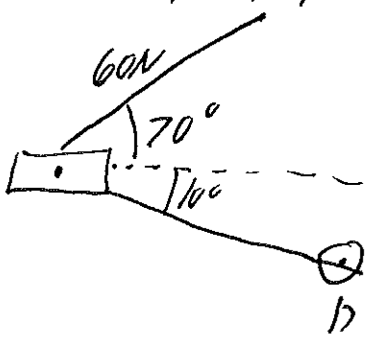
Physics Fact

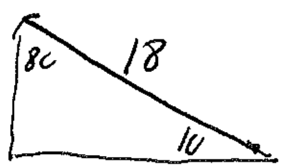


Torque $= \tau = \vec{r} \times \vec{F}$ measures tendency to rotate about origin.

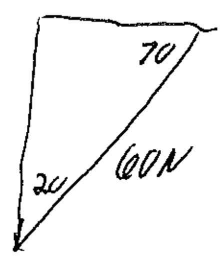
#39 Bicycle pedal pushed by foot w/ 60-N force

shaft is 18cm. Find mag of torque about P





$\vec{r} = (-18\cos 10, 18\sin 10) = (-17.727, 3.12)$



$\vec{F} = (-60\cos 70, -60\sin 70)$

$= (-20.52, -56.38)$

$\vec{F} \times \vec{r} = (-20.5, -56.4, 0) \times (-17.7, 3.12, 0)$

$= (0, 0, -1062)$ $\boxed{1062\ N \cdot cm}$

#45 P a point not on line $L = \overleftrightarrow{QR}$. Show distance from P to L is

$$d = \frac{|\vec{QR} \times \vec{QP}|}{|\vec{QR}|}$$

Pf. $= \dfrac{|\vec{QR}||QP| \sin\theta}{|QR|} = |QP| \sin\theta$