Lecture 26

Review $\vec r(u,v)$ $(u,v)\in D$ parametrized, orientable surface, unit normal

$$\vec n = \frac{\vec r_u \times \vec r_v}{|\vec r_u \times \vec r_v|} \quad , \text{ denote surface } S$$

$\vec F$ a vector field.

Surface integral of $\vec F$ over S is $\displaystyle \iint_S \vec F \cdot \overbrace{\vec n\, dS}^{d\vec S} = \iint_D \vec F(\vec r(u,v)) \cdot (\vec r_u \times \vec r_v)\, dA$

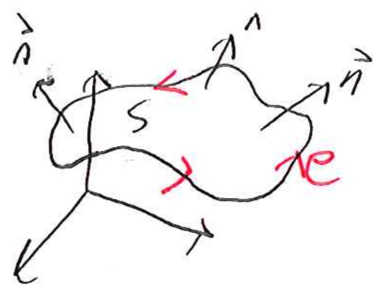
* calculates flux of $\vec F$ across S

Two important thms..

Stokes S a nice surface bounded by C w/ positive orientation
$\vec F$ a nice vector field. Then

$$\int_C \vec F \cdot d\vec r = \iint_S \text{curl}\, \vec F \cdot d\vec S$$

$$= \iint_S \text{curl}\, \vec F \cdot \vec n$$

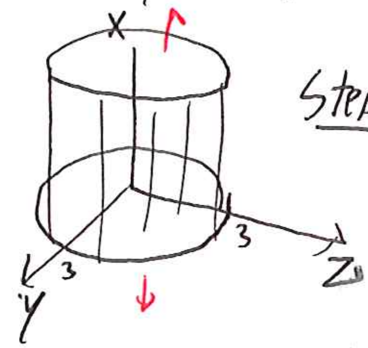


Divergence (Gauss) Thm E simple solid region, $S = \partial E$ oriented outward normal
$\vec F$ a nice vector field. Then

$$\iint_S \vec F \cdot d\vec S = \iiint_E \text{div}\, F \, dV.$$

16.9 #4 $\vec{F} = (x^2, -y, z)$ E = cylinder $y^2+z^2 \le 9$, $0 \le x \le 2$ ②

Verify divergence Thm (by calculating both integrals)



Step 1 Parametrize ∂E in 3 pieces.

Top $\vec{r}(u,v) = (2, u\cos v, u\sin v)$ $0 \le u \le 3$
$0 \le v \le 2\pi$

$r_u = (0, \cos v, \sin v)$
$r_v = (0, -u\sin v, u\cos v)$ $r_u \times r_v = (u, 0, 0)$ points up

Bottom $\vec{r}(u,v) = (0, u\sin v, u\cos v)$ $0 \le u \le 3$ $0 \le v \le 2\pi$
$r_u = (0, \sin v, \cos v)$ $r_v = (0, u\cos v, -u\sin v)$ $r_u \times r_v = (-u, 0, 0)$ down

Side $\vec{r}(u,v) = (u, 3\sin v, 3\cos v)$ $0 \le u \le 2$
$0 \le v \le 2\pi$

$r_u = (1, 0, 0)$ $r_v = (0, 3\cos v, -3\sin v)$ $r_u \times r_v = (0, 3\sin v, 3\cos v)$
points out.

Step 2 Evaluate 3 surface integrals

Top $\int_0^3 \int_0^{2\pi} (4, -u\cos v, u\sin v) \cdot (u, 0, 0) = \int_0^3 \int_0^{2\pi} 4u \, dv \, du = \int_0^3 8\pi u \, du$
$= 36\pi$

Bottom $\int_0^3 \int_0^{2\pi} (0, -u\sin v, u\cos v) \cdot (-u, 0, 0) = 0$

Side $\int_0^2 \int_0^{2\pi} (u^2, -3\sin v, 3\cos v) \cdot (0, 3\sin v, 3\cos v)$

$= \int_0^2 \int_0^{2\pi} -9\sin^2 v + 9\cos^2 v \, dv \, du$

$= 9 \int_0^2 \int_0^{2\pi} \cos(2v) = 0.$

$\boxed{\iint_S \vec{F} \cdot \vec{n} = 36\pi}$

Step 3 $\operatorname{div} F = 2x - 1 + 1 = 2x$

Do $\iiint_E 2x$ in cylindrical: $0 \le r \le 3$ $0 \le \theta \le 2\pi$ $0 \le x \le 2$

$$\int_0^{2\pi}\int_0^3\int_0^2 2x \cdot r\,dx\,dr\,d\theta = \int_0^{2\pi}\int_0^3 r x^2 \Big|_0^2 \,dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^3 4r\,dr\,d\theta = \int_0^{2\pi} 2r^2\Big|_0^3 = \int_0^{2\pi} 18\,d\theta = 36\pi$$

16.9 #8 $\vec{F} = (x^3+y^3, y^3+z^3, z^3+x^3)$

S = sphere, radius 2, center (0,0,0)! Use div Thm to calculate $\iint_S \vec{F}\cdot d\vec{S}$

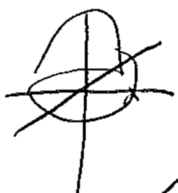
A $\operatorname{div} F = 3x^2+3y^2+3z^2$ so want $\iiint_E 3x^2+3y^2+3z^2$

Use spherical $\displaystyle\int_0^{2\pi}\int_0^{\pi}\int_0^{2} 3\rho^2 \cdot \rho^2 \sin\phi \, d\rho\, d\phi\, d\theta = \frac{384}{5}\pi$

Review #31 $\vec{F} = (x^2, y^2, z^2)$ S is paraboloid $z = 1 - x^2 - y^2$ above xy plane, pos or.

Verify Stokes

$\operatorname{curl} \vec{F} = (0,0,0)$ so $\iint \operatorname{curl} \vec{F} \cdot \vec{n} = 0$.



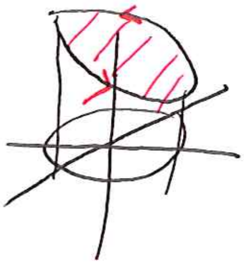
$C = (\cos t, \sin t, 0)$ $0 \le t \le 2\pi$

$$\int_C \vec{F}\cdot dr = \int_0^{2\pi} (\cos^2 t, \sin^2 t, 0)\cdot(-\sin t, \cos t, 0)$$

$$= \int_0^{2\pi} -\sin t\cos^2 t + \sin^2 t\cos t = \frac{1}{3}(\cos^3 t + \sin^3 t)\Big|_0^{2\pi}$$

$$= 0 \;//$$

16.8 #10 $\vec{F} = (2y, xz, x+y)$ $\quad \mathcal{C}$ = inter of plane $z = y+2$ w/ $x^2+y^2=1$.



Use Stokes' to evaluate $\oint_{\mathcal{C}} \vec{F}\cdot d\vec{r}$

$\mathcal{C}$ ccwise from above

$$\text{curl}\,\vec{F} = (1-x,\ 0-1,\ z-2) = (1-x,\ -1,\ z-2)$$

Parametrize surface $\vec{r}(x,y) = (x, y, y+2)$ $\quad x,y \in$ 

$\vec{r}_x = (1,0,0)$ $\quad r_y = (0,1,1)$ $\quad r_x \times r_y = (0,-1,1)$ up

By Stokes' we get

$$\iint_D \text{curl}\,\vec{F}(\vec{r}(x,y))\cdot(0,-1,1) = \iint_D (1-x,-1,y)\cdot(0,-1,1)$$

$$= \iint_D 1+y = \int_0^{2\pi}\int_0^1 (1+r\sin\theta)\,r\,dr\,d\theta$$

$$= \int_0^{2\pi} \frac{r^2}{2} + \frac{r^3}{3}\sin\theta \Big|_0^1$$

$$= \int_0^{2\pi} \frac{1}{2} + \frac{1}{3}\sin\theta = \frac{1}{2}\theta - \frac{1}{3}\cos\theta \Big|_0^{2\pi}$$

$$= \pi$$