Lecture 25

Review
- Parametrized surface $\vec{r}(u,v)$ for $(u,v) \in D$.
- $\vec{r}_u(u,v)$ and $\vec{r}_v(u,v)$ are tangent to surface
- $\vec{r}_u(u_0,v_0)$, $\vec{r}_v(u_0,v_0)$ are tangent at $\vec{r}(u_0,v_0)$

Tangent plane: $\vec{r}(s,t) = r_0(u_0,v_0) + s\, r_u(u_0,v_0) + t\, r_v(u_0,v_0)$

normal vector $\vec{n} = \vec{r}_u(u_0,v_0) \times \vec{r}_v(u_0,v_0)$

Surface integrals $f(x,y,z)$ defined on S param by $\vec{r}(u,v)$ $(u,v) \in D$ Then

$$\iint_S f(x,y,z)\,dS = \iint_D f(\vec{r}(u,v))\,|\vec{r}_u \times \vec{r}_v|\,dA$$

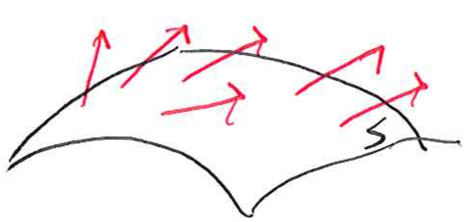
* $f = 1$ gives formula of surface area.

Oriented Surfaces
- Examples
- $\vec{n} = \dfrac{\vec{r}_u \times \vec{r}_v}{|\vec{r}_u \times \vec{r}_v|}$ unit normal vector.

- arbitrary choice for closed surfaces:

"positive orientation" = "$\vec{n}$ points out"

Problem Fluid flow, surface S, how much is flowing through S?



$\vec{F}$ = vector field

* Only component of $\vec{F}$ normal to S contributes to flow through S

* This component is $\vec{F} \cdot \dfrac{\vec{n}}{|\vec{n}|} = \vec{F} \cdot \dfrac{r_u \times r_v}{|r_u \times r_v|}$

Surface integrals of vector fields

Function is $\vec{F} \cdot \frac{\vec{n}}{|\vec{n}|}$ so $\iint_S g\, dS = \iint_D F \cdot \frac{r_u \times r_v}{|r_u \times r_v|} \cdot |r_u \times r_v|\, dA$

$\quad\;\; \overset{\shortparallel}{g}$

Conclude $\vec{F}$ continuous vector field defined on oriented surface S w/ unit normal vector $\vec{n}$, the surface integral of $\vec{F}$ over S is

$$\iint_S \vec{F} \cdot d\vec{S} = \iint_S \vec{F} \cdot \vec{n}\, dS \quad \leftarrow \text{Notation}$$

$$= \iint_D \vec{F} \cdot (\vec{r}_u \times \vec{r}_v)\, ds \quad \leftarrow \text{How to calculate}$$

a.k.a. flux of $\vec{F}$ across S

Ex S is helicoid $\vec{r}(u,v) = \langle u\cos v, u\sin v, v\rangle$ $0 \le u \le 1$ $0 \le v \le \pi$ w/ upward orientation

$\vec{F} = (z, y, x)$ Find flux of $\vec{F}$ across S

Answer $r_u = (\cos v, \sin v, 0)$ $r_v = (-u\sin v, u\cos v, 1)$ $r_u \times r_v = (\sin v, -\cos v, u)$

↑ points "up" for $0 \le u \le 1$

$F(\vec{r}(u,v)) = (v, u\sin v, u\cos v)$

$$\iint \vec{F} \cdot \vec{n} = \int_0^1 \int_0^\pi (v, u\sin v, u\cos v) \cdot (\sin v, -\cos v, u)\, dv\, du$$

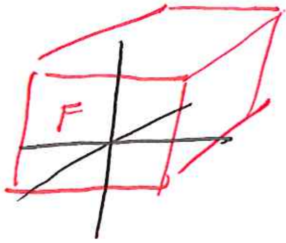
$$= \int_0^1 \int_0^\pi v\sin v - u\sin v\cos v + u^2\cos v\, dv\, du$$

$$= \int_0^1 \sin v - v\cos v - \frac{u}{2}\sin^2 v + u^2 \sin v \Big|_0^\pi du$$

$$= \int_0^1 \pi\, du = \pi$$

16.7 #29 $\vec{F} = (x, 2y, 3z)$ S = cube vertices $(\pm 1, \pm 1, \pm 1)$

Evaluate $\iint_S \vec{F} \cdot dS$ w/ outward orientation



Six surfaces to parametrize. But can simplify!

F: $r_y \times r_z = (1,0,0)$

$F = (1, 2y, 3z)$ $\quad -1 \le y \le 1 \quad -1 \le z \le 1$ etc..

Ex $\vec{u}(x,y,z)$ = temperature. Heat flow is $\vec{F} = -k \nabla \vec{u}$

conductivity

Then $\iint \vec{F} \cdot d\vec{S} = -k \iint \nabla u \cdot dS$ measures rate of heat flow across surface

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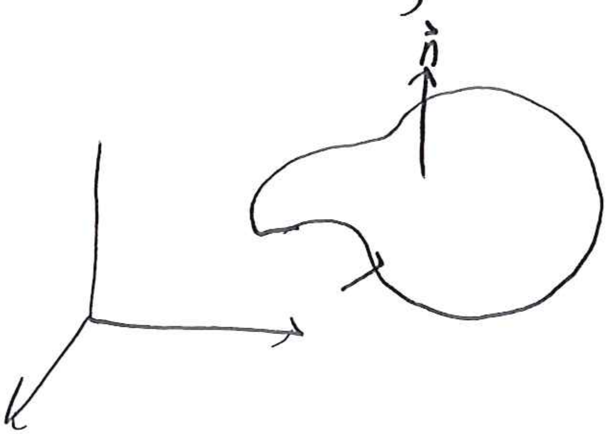
Stokes' Thm and Gauss' Thm

Stokes Thm S oriented piecewise smooth surface bounded by simple, closed, piecewise smooth boundary curve w/ pos. orientation

$\vec{F}$ a vector field w/ components having continuous partials

Then

$$\iint_S \text{curl}\,\vec{F} \cdot d\vec{S} = \int_C \vec{F} \cdot d\vec{r}$$



walk around, head pts toward $\vec{n}$, surface on left

Motivation Work done by $\vec{F}$ around curve = total of circulation.

Special Case S lies in xy plane so unit normal is $(0,0,1)$, we get

$$\oint_{C} \vec{F}\cdot d\vec{r} = \iint_{S} \operatorname{curl}\vec{F}\cdot(0,0,1) = \iint_{S} \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}$$

* Green's Thm is a special case of Stokes' *

16.8 #2

Ex Use Stokes' Thm to evaluate $\iint_{S} \operatorname{curl}\vec{F}\cdot dS$

$\vec{F} = (x^2 \sin z, y^2, xy)$

S = paraboloid $z = 1 - x^2 - y^2$ above x-y plane oriented up.

16.8 #10

Ex Use Stokes' Thm to evalue $\oint_{C} \vec{F}\cdot d\vec{r}$, C ccwise viewed from above.

$\vec{F} = (2y, xz, x+y)$

C = curve of intersection of $z = y+2$, cylinder $x^2+y^2=1$

Gauss (Divergence) Thm

E simple solid region in $\mathbb{R}^3$ w/ boundary S oriented positively

$\vec{F}$ a vector field w/ continuous partials

Then $$\iint_{S} \vec{F}\cdot dS = \iiint_{E} \operatorname{div}\vec{F}\cdot dV$$