Lecture 24

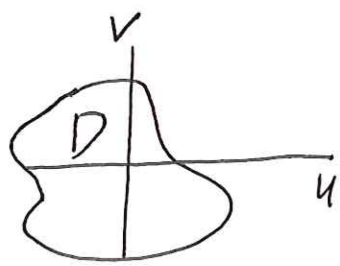
Review
1. $\vec{F}(x_1, x_2, \dots, x_n) = (F_1(x_1, \dots, x_n), \dots, F_n(x_1, \dots, x_n))$ vector field, $\operatorname{div} \vec{F} = \nabla \cdot \vec{F} = \sum_{i=1}^{n} \frac{\partial F_i}{\partial x_i}$
2. $\vec{F}(x,y,z) = (P(x,y,z), Q(x,y,z), R(x,y,z))$ $\quad \operatorname{curl} \vec{F} = \nabla \times \vec{F}$
3. On simply conn regions, $\operatorname{curl} \vec{F} = \vec{0} \Leftrightarrow \vec{F}$ cons.
4. $\operatorname{curl}(\nabla f) = \vec{0}$, $\operatorname{div}(\operatorname{curl} F) = 0$.

Idea: Generalize line integrals, Green's Thm, to surfaces.

Parametric Surfaces

Recall $\vec{r}(t) = (x(t), y(t), z(t))$ $a \le t \le b$ gives space curve.

Suppose

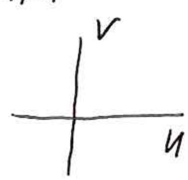


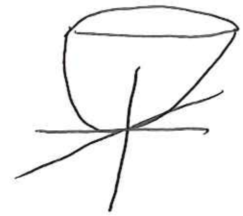
$\vec{r}(u,v) = (x(u,v), y(u,v), z(u,v))$ as $(u,v) \in D$ gives parametrized surface

Examples

1. Graph of $z = f(x,y)$ $\quad \vec{r}(u,v) = (u, v, f(u,v))$

$$\vec{r}(u,v) = (u, v, u^2 + v^2)$$





compare $\vec{r}(t) = (t, f(t))$ for $y = f(x)$

2. Plane containing $\vec{r}_0$ and two nonparallel vectors $\vec{a}$ & $\vec{b}$

$$\vec{r}(u,v) = \vec{r}_0 + u\vec{a} + v\vec{b}$$

Ex $\quad \vec{r}(u,v) = (3 + 2u + 6v, 1 - u - v, 2 + u)$

$$= (3,1,2) + u(2,-1,1) + v(6,-1,0)$$

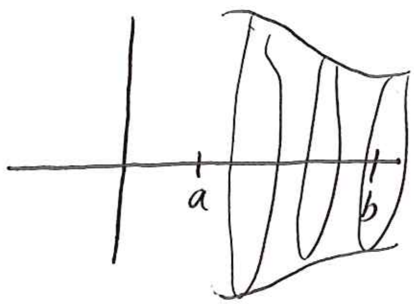
Compare line: $\vec{r}(t) = \vec{r}_0 + t\vec{v}$

3. Sphere radius a: $\vec{r}(\phi, \theta) = (a\sin\phi\cos\theta,\ a\sin\phi\sin\theta,\ a\cos\phi)$

$0 \le \phi \le \pi$
$0 \le \theta \le 2\pi$

4. $\vec{r}(\theta, z) = (3\cos\theta,\ 3\sin\theta,\ z)$

5. Revolve $y = f(x) > 0$ about x axis



$x = x$
$y = f(x)\cos\theta$
$z = f(x)\sin\theta$

6. Maple for complicated surfaces. Point out curves $\vec{r}(u_0, \vec{v})$, $\vec{r}(u, v_0)$

Tangent Planes

Observe $\vec{r}(u_0, v)$ and $\vec{r}(u, v_0)$ both curves in surface passing through $\vec{r}(u_0, v_0)$.

$\vec{r} = (x(u,v), y(u,v), z(u,v))$ $\quad \vec{r}_u = \left(\frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}, \frac{\partial z}{\partial u}\right)$

$\vec{r}_v = \left(\frac{\partial x}{\partial v}, \frac{\partial y}{\partial v}, \frac{\partial z}{\partial v}\right)$ both tangent.

* If $\vec{r}_u(u_0, v_0) \times \vec{r}_v(u_0, v_0) \neq 0$ then it is a normal vector to tangent plane

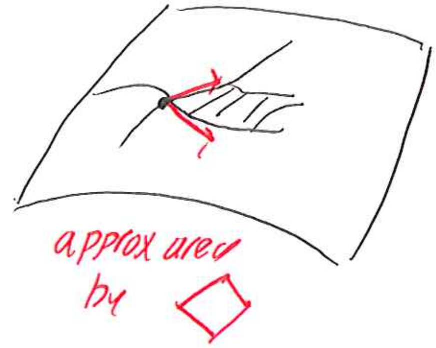
Ex 1 $\vec{r}(u,v) = (u^2+1, v^3+1, u+v)$ at $(5,2,3)$. Find tang plane in both parametric and $ax+by+cz=d$ form

Ex 2 Repeat for $\vec{r}(u,v) = (u^2, 2v\sin v, u\cos v)$ at $u=1, v=0$

Surface Area

Recall For arc length, $\sqrt{x'(t)^2+y'(t)^2}\,dt$ was approximation.

For area $|\vec{r}_u \times \vec{r}_v|$ is area of parallelogram



Thm Let $\vec{r}(u,v) = (x(u,v), y(u,v), z(u,v))$ $(u,v)\in D$ be a smooth, param. surface S.

The surface area of S is $\iint_D |\vec{r}_u \times \vec{r}_v|\, dA$.

Ex $\vec{r}(u,v) = (u\cos v, u\sin v, v)$ $0 \le u \le 1$, $0 \le v \le \pi$. Find S.A (surface on maple)

A $\vec{r}_u = (\cos v, \sin v, 0)$ $\quad \vec{r}_v = (-u\sin v, u\cos v, 1)$

$\vec{r}_u \times \vec{r}_v = (\sin v, -\cos v, u\cos^2 v + u\sin^2 v) = (\sin v, -\cos v, u)$

$|\vec{r}_u \times \vec{r}_v| = \sqrt{\sin^2 v + \cos^2 v + u^2} = \sqrt{1+u^2}$

Want $\displaystyle\int_0^{\pi}\int_0^1 \sqrt{1+u^2}\,du\,dv \underset{\text{TABLES}}{=} \pi \cdot \left(\frac{u}{2}\sqrt{1+u^2} + \frac{1}{2}\ln\left|u+\sqrt{u^2+1}\right|\right)\Big|_0^1$

$= \pi/2\left(\sqrt{2} + \ln(1+\sqrt{2})\right)$

Ex $z = f(x,y)$ $(x,y)\in D$. Find formula for S.A

Surface Integrals

Review $f(x,y)$ on C, $\int_C f(x,y)\,dS = \int f(r(t)) \cdot |\vec{r}\,'(t)|\,dt$
param by $\vec{r}(t)$

Special Case: $\int_C \vec{F} \cdot d\vec{r} = \int \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)\,dt$

Goal: Parallel devel. for surface ints

Def S surface param by $\vec{r}(u,v)$, $(u,v) \in D$, $f(x,y,z)$ a function. The <u>surface integral of f over S</u> is:

$$\iint_S f(x,y,z)\,dS = \iint_D f(\vec{r}(u,v))\,|\vec{r}_u \times \vec{r}_v|\,dA.$$

Ex 1 $f(x,y,z) = 1$, just get surface area.

Ex 2 S is cone $y = \sqrt{x^2+y^2}$ $0 \le y \le 5$. Find $\iint_S y^2 z^2\,dS$

Ex 3 $\iint_S xz\,dS$ S is plane $2x+2y+z=4$ in 1st octant.

Procedure

1. Parametrize S as $\vec{r}(u,v)$ $(u,v) \in D$
2. Compute $f(\vec{r}(u,v))$, $\vec{r}_u$, $\vec{r}_v$, $|\vec{r}_u \times \vec{r}_v|$
3. Evaluate

Oriented Surfaces

- Examples
- $\vec{n} = \dfrac{\vec{r}_u \times \vec{r}_v}{|\vec{r}_u \times \vec{r}_v|}$ unit normal
- arbitrary choice for closed surfaces

"positive orientation" = "$\vec{n}$ points outward"

Surface Integrals of Vector Fields

$\vec{F}$ a force field, $\vec{r}(u,v)$ a surface, $\vec{n} = \cancel{|\vec{r}_u \times \vec{r}_v|}\ \dfrac{r_u \times r_v}{|r_u \times r_v|}$

$\vec{F} \cdot \vec{n}$ = component of $\vec{F}$ $\perp$ to S,

measures "flux" across S

Def $\vec{F}$ a continuous vector field defined on oriented surface S, unit normal $\vec{n}$.

The surface integral of $\vec{F}$ over S is

$$\iint_S \vec{F} \cdot d\vec{S} = \iint_S \vec{F} \cdot \vec{n}\, dS$$

To evaluate:

$$\iint_S \vec{F} \cdot \vec{n}\, dS = \iint_D \vec{F}(r(u,v)) \cdot \frac{r_u \times r_v}{|r_u \times r_v|}\, |r_u \times r_v|\, dA$$

$$= \iint_D \vec{F}(\vec{r}(u,v)) \cdot (\vec{r}_u \times \vec{r}_v)\, dA$$