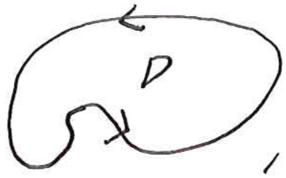
Lecture 23

$\mathcal{C}$ simple, closed, ccwise 

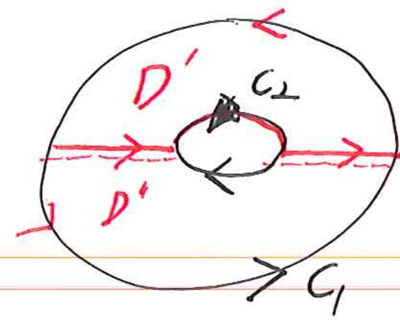
, D region enclosed, $\vec{F}(x,y) = (P(x,y), Q(x,y))$

Green's Thm $\qquad \displaystyle\oint_{\mathcal{C}} \vec{F}\cdot d\vec{r} = \iint_D \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\, dA$

Example $\mathcal{C}$ boundary enclosed by $y = x^2$, $x = y^2$, ccwise. Find

$$\int_{\mathcal{C}} (y + e^{\sqrt{x}})\,dx + (2x + \cos(y^2))\,dy.$$

Rmk $\iint$ is much easier.

Generalize Suppose region has a hole 

Green applies to D', D'' so

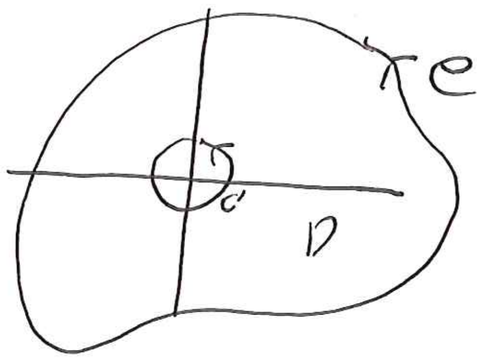
$$\iint_D \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = \iint_{D'} + \iint_{D''} = \int_{\partial D'} F + \int_{\partial D''} F \quad \text{cancels!}$$

$$= \int_{\mathcal{C}_1} P\,dx + Q\,dy + \int_{\mathcal{C}_2} P\,dx + Q\,dy$$

* Note C_1, C_2 have region on left.

Ex $\vec{F}(x,y) = \dfrac{-yi + xj}{x^2+y^2}$. Show $\oint_C \vec{F}\cdot dr = 2\pi$ for any simple closed curve about origin

Pf Notice $\dfrac{\partial Q}{\partial x} = \dfrac{\partial P}{\partial y}$



$\oint_C \vec{F} - \oint_{C'} \vec{F} = \iint_D 0 = 0$

C': $(a\cos t, a\sin t)$ ← calculate

$\int_0^{2\pi}$ ~~$-a\,a$ $-a\cos t$~~

* but $\dfrac{\partial Q}{\partial x} = \dfrac{\partial P}{\partial y}$, so why is this not indep of path?

16.5 Curl & Divergence

Notation Think of ∇ as operator $\left(\dfrac{\partial}{\partial x}, \dfrac{\partial}{\partial y}, \dfrac{\partial}{\partial z}\right) = \dfrac{\partial}{\partial x}\vec{i} + \dfrac{\partial}{\partial y}\vec{j} + \dfrac{\partial}{\partial z}\vec{k}$

Rmk $\nabla f = \left(\dfrac{\partial f}{\partial x}, \dfrac{\partial f}{\partial y}, \dfrac{\partial f}{\partial z}\right)$ as before

Def Let $\vec{F}(x,y,z) = (P(x,y,z), Q(x,y,z), R(x,y,z))$ be vector field on $\mathbb{R}^3$

The curl, $\text{curl}\,\vec{F} = \nabla \times F = \left(\dfrac{\partial}{\partial x}, \dfrac{\partial}{\partial y}, \dfrac{\partial}{\partial z}\right) \times (P, Q, R)$ so

$$\boxed{\text{curl}\,F = \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}, \frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}, \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)}$$

Ex $\vec{F}=(xy^2z^2, x^2y, xy+z^3)$ Find curl $\vec{F}$

curl $\vec{F}=(x-0, 2xy^2z-y, 2xy-2xyz^2)$

Thm Suppose $f(x,y,z)$ has continuous second order partials. Then

$$\text{curl}(\nabla f)=0$$

Pf $\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)\times\left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right)=(0,0,0)$ by Clairaut.

Restate Conservative vector fields have curl 0.

Rmk $\vec{F}=(P(x,y), Q(x,y), 0)$ then curl $F=\left(0,0,\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right)$

Thm Suppose $\vec{F}(x,y,z)$ defined on simply conn region and curl $\vec{F}=0$. Then $\vec{F}$ is conservative.

Rmk Generalizes 2-dim

Ex $\vec{F}=(e^{yz}, xze^{yz}, xye^{yz})$ Is $\vec{F}$ conservative? If so find a potential Funct.

Question What does curl $\vec{F}$ measure?

* Points $\perp$ to rotation, length is rotation.

* curl $F=0$ called irrotational

Ex $F(x,y)=(-y,x)$ curl $=(0,0,2)$

Def $\vec{F}=(P,Q,R)$, the <u>divergence</u> is

$$\operatorname{div}\vec{F} = \nabla\cdot\vec{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}$$

<u>Exercise</u> $\operatorname{div}(\operatorname{curl}\vec{F}) = 0$

<u>Summary</u>

f a function, ∇f is a vector field

$\vec{F}=(P,Q,R)$ a vector field, $\operatorname{curl}\vec{F}$ also a vector field

$\vec{F}=(P,Q,R)$ a vector field $\operatorname{div}\vec{F}$ is a function

See #17 sec 16.5

<u>Q</u> What does divergence measure?

<u>incompressible</u>

<u>Problem</u> Is there a vector field $\vec{G}$ so

$$\operatorname{curl}\vec{G} = (x\sin y, \cos y, z-xy)?$$

divergence is $\sin y - \sin y + 1 \neq 0$ so no!

<u>Problem</u> $F=(\ln(2y+3z), \ln(x+3z), \ln(x+2y))$

Find $\operatorname{curl} F$ & $\operatorname{div} F$.