Lecture 22

Review $\mathcal{C}$ param by $\vec{r}(t)$, $a \le t \le b$, $\vec{F}$ a vector field

- $\int_{\mathcal{C}} \vec{F} \cdot d\vec{r} = \int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)\,dt$ line integral

Facts • Piecewise smooth $\mathcal{C} = \mathcal{C}_1$ then C_2.

$\int_{\mathcal{C}} = \int_{\mathcal{C}_1} + \int_{\mathcal{C}_2}$

- $\int_{-\mathcal{C}} \vec{F} \cdot dr = -\int_{\mathcal{C}} \vec{F} \cdot d$

Def $\vec{F}$ is a conservative vector field if $\vec{F} = \nabla f$. f potential function

Thm $\mathcal{C}$ smooth given by $\vec{r}(t)$, $a \le t \le b$
f differentiable with ∇f continuous on $\mathcal{C}$. Then

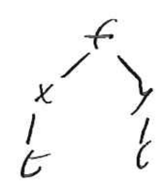
$$\int_{\mathcal{C}} \nabla f \cdot d\vec{r} = f(\vec{r}(b)) - f(\vec{r}(a))$$

* compare w/ FTOC

Proof Let $\vec{r}(t) = \langle x(t), y(t) \rangle$

$$\int_a^b \nabla f(r(t)) \cdot r'(t)\,dt = \int_a^b \frac{\partial f}{\partial x} \cdot \frac{\partial x}{\partial t} + \frac{\partial f}{\partial y} \cdot \frac{\partial y}{\partial t}\,dt$$

$$= \int_a^b \frac{d}{dt} f(\vec{r}(t))\,dt \text{ by chain rule}$$



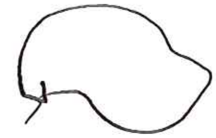
$$= f(\vec{r}(b)) - f(\vec{r}(a)) \text{ by FTOC} \quad //$$

Cor If $\mathcal{C}_1$ and C_2 have same endpoints then

$$\int_{\mathcal{C}_1} \nabla f \cdot dr = \int_{C_2} \nabla f \cdot d$$

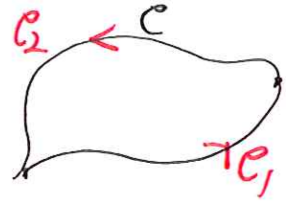
* Say line integrals of conservative vector fields are <u>independent of path</u>.

<u>Thm</u> $\int_C \vec{F}\cdot d\vec{r}$ is independent of path if & only if $\int_C \vec{F}\cdot d\vec{r} = 0$ for all closed curves C.



← closed curve.

<u>Pf</u>



write $C = C_1$ then C_2

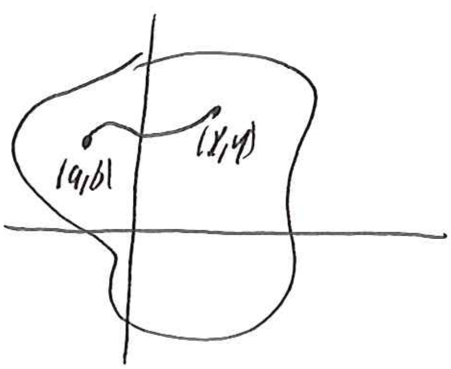
so C_1 and $-C_2$ have same endpoints

<u>Q</u> Conservative vector fields have path indep. What about reverse?

<u>Thm</u> Suppose $\vec{F}$ is continuous on open, connected region D.

If $\int_C \vec{F}\cdot d\vec{r}$ is ind of path in D then $\exists f$ so $\nabla f = \vec{F}$.

<u>Proof</u> Pick $(a,b) \in D$ Define $f(x,y) = \int_{(a,b)}^{(x,y)} \vec{F}\cdot d\vec{r}$ <u>any path</u> $(a,b) \to (x,y)$



Hard work: $\Rightarrow \nabla f(x,y) = \vec{F}$. //

<u>Question</u> Given $\vec{F}(x,y) = (P(x,y), Q(x,y))$, is $\vec{F}$ conservative?

<u>Rmk</u> If $\vec{F} = \nabla f$ then $\frac{\partial f}{\partial x} = P$, $\frac{\partial f}{\partial y} = Q$ so

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}.$$

Thm $\vec{F}(x,y) = (P(x,y), Q(x,y))$ on a simply connected region D

Suppose $\dfrac{\partial P}{\partial y} = \dfrac{\partial Q}{\partial x}$ on D. Then $\vec{F}$ is conservative

Pf Later

Ex $\vec{F}(x,y) = (3+2xy,\ x^2-3y^2)$. Show $\vec{F}$ is cons. Find f

$\dfrac{\partial P}{\partial y} = 2x = \dfrac{\partial Q}{\partial x}$. If $3+2xy = \cancel{\partial p}\ \dfrac{\partial f}{\partial x}$ then $f = 3x + x^2y + C(y)$

$x^2 - 3y^2 = \dfrac{\partial f}{\partial y}$ then $f = x^2y - y^3 + C(x)$

$$\boxed{f(x,y) = x^2y + 3x - y^3}$$

Ex $\vec{F}$ as above. Find $\int_C \vec{F}\cdot d\vec{r}$ C is $(e^t \sin t, e^t \cos t)$ $0 \le t \le \pi$.

Read about conservation of energy

Problem $\vec{F} = (x^3, y^3)$. Find work done moving object from $(2,1)$ to $(5,3)$

#35 $\vec{F} = \dfrac{-y\,\vec{i} + x\,\vec{j}}{x^2+y^2}$

a. show $\dfrac{\partial P}{\partial y} = \dfrac{\partial Q}{\partial x}$

b. show not independent of path   Explain!

16.4 Green's Thm

closed int. $[a, b]$, boundary is $\{a\} \cup \{b\}$

Recall FTOC $\int_a^b F'(x)\,dx = F(b) - F(a)$

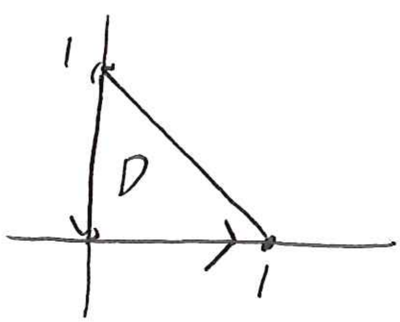
Thm C positively oriented, simple, closed curve bounding region D.

Suppose $P(x,y)$, $Q(x,y)$ have continuous partials on open region containing D.

Then
$$\int_C P\,dx + Q\,dy = \iint_D \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\; dA$$

- Define pos or, simple
- Notation $\oint P\,dx + Q\,dy$
- Can write $\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA = \int_{\partial D} P\,dx + Q\,dy$

Ex $\int_C x^4\,dx + xy\,dy$ C triang $(0,0)$ to $(1,0)$ to $(0,1)$



Green's Thm
$$= \iint_D y\,dA = \int_0^1 \int_0^{1-x} y\,dy\,dx$$
$$= \int_0^1 \frac{(1-x)^2}{2}\,dx = -\frac{1}{6}(1-x)^3\Big|_0^1$$
$$= 1/6$$

* Do as 3 line integrals & compare

Special Case Suppose $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 1$ Then

$$\oint P\,dx + Q\,dy = \iint_D 1 = \text{area of } D.$$

Ex $(P,Q) = (0,x)$
$= (-y,0)$
$= (-\tfrac{1}{2}y, \tfrac{1}{2}x)$

$$\text{Area} = \oint x\,dy = -\oint_C y\,dx = \tfrac{1}{2}\oint_C x\,dy - y\,dx$$

Ex Area of ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ planimeter!

$\vec{r}(t) = (a\cos t, b\sin t)$ using 3rd formula

Question
What if $D =$    slice it!